Order convergence and distance on Łukasiewicz-Moisil algebras

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Dedicated to the memory of Professor Helmut Thiele.

Abstract

The paper develops a study of order convergence in Łukasiewicz-Moisil algebras. An axiomatical notion of distance (covering the pointwise and the Heyting distances) is provided, together with an associated notion of Cauchy sequence. Under natural hypotheses, it is proven the existence of Cauchy completions. It is analysed the connection to Boolean algebras along the canonical adjunction. The special class of proper LM_m-algebras with Łukasiewicz distance is also investigated. Finally, we provide characterizations for the Cauchy completions corresponding to some particular class of axiomatic distances.

Keywords: Łukasiewicz-Moisil algebra, order convergence, distance, Cauchy completion, proper algebra, MV_n-algebra.

1 Introduction

Łukasiewicz-Moisil algebras (LM-algebras for short) were introduced by Moisil back in the early fourties, in [18] (the 3 and 4-valued versions) and in [19] (the general, n-valued version), under the name "Łukasiewicz algebras", as an algebraic counterpart of the corresponding multi-valued logics of Łukasiewicz. These structures generalize Boolean algebras in the sense of not forcing elements, regarded as truth values, to satisfy the *tertium non datur* priciple; but still allowing, for each element, a total hierarchy of n Boolean (or Chryssippian) nuances, to which it is reducible. The study of these structures was stimulated both by their logical and technical applications (to electric circuits). There exists a significant amount of literature dedicated to Łukasiewicz-Moisil algebras - the monography [2] collects an important part of these results.

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Unlike other structures derived from logic that generalize Boolean algebras (like residuated lattices or Boolean algebras), Łukasiewicz-Moisil algebras share with Boolean algebras (and with MV-algebras, some other "Lukasiewicz's algebras") the symmetrical structure. Consequently, diverse "implicative" LM-operators, with their associated "iff" operators, provide, by dualization, "metrical" operators, that is binary commutative operators with properties similar to those of classical distances - hence the virtual topological dimension of LM-algebras, with fundamentally different agenda then the logical dimension. Metrical and topological study of logic-derived algebras has important antecedents. Different kinds of convergence in Boolean algebras were treated in [16] and [25]; also, MV-algebraic convergence made the subject of [13] and [5]. All these were paralleled by convergence and completion results for traditionally metrical and "completable" structures, like distributive lattices [3], or lattice-ordered groups [23], [25], [4].

The present work comes to join the above series of papers, by studying order convergence in LM-algebras. Due to their mentioned symmetrical structure, LM-algebras provide a "good colaboration" between distance and order, which makes that the notion of Cauchy sequence be very natural in the context of order convergence. The paper is structured as follows:

Section 2 establishes the notations and reminds some basic facts about LM-algebras.

Section 3 initiates a study of LM order convergence and relates it to some axiomatical notion of distance d, general enough to capture both the pointwise distance (d_P) and the Heyting distance (d_H). The Cauchy property of sequneces is defined in terms of the distance d.

Section 4 is dedicated to Cauchy completions. First, we provide some Cauchy completion criteria for classes of LM-algebras equipped with distances - these criteria show the agreement of different definitions of order Cauchy completion in the literature. Then, assuming that the discussed class is a variety (including the cases of polinomially-defined distances), we construct the Cauchy completion in that class.

Section 5 relates LM-algebras to their Boolean centers w.r.t. order convergence, along the already established adjunction relation [12] and its axled extension [26].

In Section 6 the class of *proper* LM_m-algebras is considered. The convergence and the Cauchy completions w.r.t. Łukasiewicz distance are investigated.

Finally, in Section 7, we provide some characterizations for a special class of Cauchy completions using a representation of LM_m-algebras as sequences of Boolean ideals.

2 Preliminaries

Given a lattice $(L, \vee, \wedge)$, $\leq$ denotes its induced partial order and $0, 1$ the least and, respectively, greatest element (if it exists). Also:

• for $a, b \in L$, $[a, b]$ denotes all the elements placed between a and b in the sense of order;

• if exist, family suprema and infima are denoted using $\bigvee$ and $\bigwedge$; whenever the lattice in which they are considered is not clear, we use superscripts (for instance, in the lattice L, we write $\bigvee^L$ and $\bigwedge^L$);

• if $A \subseteq L$, $Lb(A)$ denotes the lower bounds of A, while $Ub(A)$ denotes the upper bounds of A.

During this paper, we shall deal with countable sequences $(c_n)_{n \in \mathbb{N}}$ of elements from the underlying sets of diverse algebraic structures (enriched lattices), which will be briefly denoted $(c_n)_n$. If $(c_n)_n$ is a sequence from a lattice, we denote by:

• $(c_n)_n \downarrow$, the fact that $(c_n)_n$ is decreasing;

• $(c_n)_n \uparrow$, the fact that $(c_n)_n$ is increasing;

• $(c_n)_n \uparrow x$, the fact that $(c_n)_n$ is increasing and $\exists \bigvee_n c_n = x$.

• $(c_n)_n \downarrow x$, the fact that $(c_n)_n$ is decreasing and $\exists \bigwedge_n c_n = x$.

Whenever the lattice L is not clear from the context, we use subscripts - for example, $(c_n)_n \uparrow_L x$, $(c_n)_n \downarrow_L 0$, $Lb_L(A)$ etc.

Definition 2.1 A structure $(L, \vee, \wedge, -, 0, 1)$ is called *dual Heyting algebra* if the following hold:
(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;
(2) (the residuation property) $x \leq y \vee z$ iff $x - y \leq z$.

Lemma 2.1 If L is a dual Heyting algebra, then the following hold (for each $x, y, z, u \in L$ I a set, and $(x_i)_{i\in I}$, $(y_i)_{i\in I}$ families from L):
(1) $-$ is increasing on the first and decreasing on the second argument;
(2) The operation $-$ is uniquely determined by the lattice structure;
(3) $x - y = 0$ iff $x \leq y$;
(4) $0 - x = 0$; $x - 0 = x$;
(5) $x - y \leq z$ iff $x - z \leq y$;
(6) $x - (y \vee z) = (x - y) - z = (x - z) - y$;
(7) $x \leq (x - y) \vee y$;
(8) $(x \vee y) - z \leq x \vee (y - z)$;
(9) $(x \vee y) - (z \vee u) \leq (x - z) \vee (y - u)$;
(10) $x - y \leq (x - z) \vee (z - y)$;
(11) $(x - y) - (z - y) \leq x - z$; $(y - x) - (y - z) \leq z - x$;
(12) If $\bigwedge_{i\in I} x_i$ exists, then so does $\bigwedge_{i\in I}(x \vee x_i)$ and is equal to $x \vee \bigwedge_{i\in I} x_i$;
(13) If $\bigwedge_{i\in I} x_i$ exists, then so does $\bigvee_{i\in I}(x - x_i)$ and is equal to $x - \bigwedge_{i\in I} x_i$;
(14) If $\bigvee_{i\in I} x_i$ exists, then so does $\bigvee_{i\in I}(x_i - x)$ and is equal to $\left(\bigvee_{i\in I} x_i\right) - x$;
(15) If $\bigvee_{i\in I} x_i$, $\bigvee_{i\in I} y_i$, and $\bigvee_{i\in I}(x_i - y_i)$ exist, then

$$\bigvee_{i\in I} x_i - \bigvee_{i\in I} y_i \leq \bigvee_{i\in I}(x_i - y_i) ;$$

(16) If $\bigwedge_{i\in I} x_i$, $\bigwedge_{i\in I} y_i$, and $\bigvee_{i\in I}(x_i - y_i)$ exist, then

$$\bigwedge_{i\in I} x_i - \bigwedge_{i\in I} y_i \leq \bigvee_{i\in I}(x_i - y_i) ;$$

(17) L has the 0-sphere property, in that whenever $(x_n)_n \downarrow 0$ and $(y_n)_n \downarrow 0$, we have $(x_n \vee y_n)_n \downarrow 0$.

Proof:
We only prove (17) (for the other points we refer the reader to [27]). Let $(x_n)_n \downarrow 0$ and $(y_n)_n \downarrow 0$ and let $t \in L$ such that, for each n, $t \leq x_n \vee y_n$; we want to show that $t = 0$. Let m be a natural number. Since $(y_n)_n$ is decreasing, we have that $\bigwedge_{n\geq m} y_n = 0$. For each $n \geq m$, $t - x_n \leq y_n$, so

$$t - x_m = t - x_n \leq y_n \ , \ \text{hence} \ \ t - x_m \leq \bigwedge_n y_n = 0 \ .$$

So, by point (2), $t \leq x_m$; and this happens for each m; thus $t \leq \bigwedge_m x_m = 0$, that is $t = 0$.
q.e.d.

From now one, all throughout the paper, m is a fixed strictly positive number.

Definition 2.2 An *m-valued Lukasiewicz-Moisil algebra (with negation)*, LM_m for short, is a structure of the form $(L, \vee, \wedge, ^-, (\varphi_i)_{i\in\{0,\ldots,m-1\}})$ such that:
(1) $(L, \vee, \wedge, ^-, 0, 1)$ is a *de Morgan algebra*, that is a bounded distributed lattice with a decreasing involution — satisfying the de Morgan property $\overline{x \vee y} = \overline{x} \wedge \overline{y}$;
(2) For each $i \in \{0, \ldots, m-1\}$, $\varphi_i : L \longrightarrow L$ is a lattice endomorphism;
(3) For each $i \in \{0, \ldots, m-1\}, x \in L$, $\varphi_i(x)$ is complemented by $^-$, that is $\varphi_i(x) \vee \overline{\varphi_i(x)} = 1$,

$\varphi_i(x) \wedge \overline{\varphi_i(x)} = 0$;
(5) For each $i, j \in \{0, \ldots, m-1\}$, $\varphi_i \circ \varphi_j = \varphi_j$;
(6) For each $i \leq j \in \{0, \ldots, m-1\}$, $\varphi_i \leq \varphi_j$;
(7) For each $i \in \{0, \ldots, m-1\}, x \in X$, $\varphi_i(\overline{x}) = \overline{\varphi_{m-i-1}(x)}$.
(8) (Moisil's determination principle)
$[\forall i \in \{0, \ldots, m-1\},\ \varphi_i(x) = \varphi_i(y)]$ implies $x = y$.

For the properties of LM_ms we refer the reader to [2]. For each LM_m, L, we define its *Boolean center*,

$$C(L) = \{x \in L \ / \ \varphi_i(x) = x, \ \ \forall i \in \{0, \ldots, m-1\}\ \} \ .$$

Lemma 2.2 Let L be a LM_m. Then the following hold:
(1) $C(L)$, with pointwise defined Boolean operations, is a Boolean algebra;
(2) $x \in C(L)$ iff $\exists i \in \{0, \ldots, m-1\}$, $\varphi_i(x) = x$ iff $\exists i \in \{0, \ldots, m-1\}, y \in Y$, $\varphi_i(y) = x$;
(3) for each $i \in \{0, \ldots, m-1\}$, $\varphi_i\left(\bigvee_{k \in K} x_k\right) = \bigvee_{k \in K} \varphi_i(x_k)$ whenever $\bigvee_{k \in K} x_k$ exists; and $\varphi_i\left(\bigwedge_{k \in K} x_k\right) = \bigwedge_{k \in K} \varphi_i(x_k)$ whenever $\bigwedge_{k \in K} x_k$ exists;
(4) $\overline{\bigvee_{k \in K} x_k} = \bigwedge_{k \in K} \overline{x_k}$ whenever $\bigvee_{k \in K} x_k$ exists; and $\overline{\bigwedge_{k \in K} x_k} = \bigvee_{k \in K} \overline{x_k}$ whenever $\bigwedge_{k \in K} x_k$ exists;
(5) $[\forall i \in \{0, \ldots, m-1\},\ \varphi_i(x) \leq \varphi_i(y)]$ iff $x \leq y$.

Lemma 2.3 (1) LM_m is a variety of algebras.
(2) Let L be a LM_m. Define $x - y = x \wedge \bigvee_{i \in \{0, \ldots, m-1\}}(\varphi_i(x) \wedge \overline{\varphi_i(y)})$. Then $(L, \vee, \wedge, -, 0, 1)$ is a dual Heyting algebra.

3 Distance and order convergence

We fix a LM_m algebra L. If we consider L only as a partially ordered set with the latice order $\leq$, we have a classical notion of order topology:

Definition 3.1 A sequence $(x_n)_n$ from L is said to order-converge (o-converge) to $x \in L$, denoted $(x_n)_n \longrightarrow x$, if there exist $(s_n)_n \uparrow x$, $(t_n)_n \downarrow x$, such that

$$(\forall n \in I\!N) s_n \leq x_n \leq t_n \ .$$

Notice that the above convergence, being preserved to taking subsequences, indeed defines a topology (the order topology) by chosing the closed sets to be those that contain the limit of a sequence whenever they include the sequence. The next lemma shows that order convergence is a natural extension of countable suprema and infima saturation.

Lemma 3.1 Let $x \in L$ and $(x_n)_n \downarrow$ be a sequence from L such that, for each $n \in I\!N$, $x \leq x_n$. Then the following are equivalent:
(1) $(x_n)_n \downarrow x$;
(2) $(x_n)_n \longrightarrow x$.

Proof:
"(2) implies (1)" is obvious.
"(1) implies (2)": $(x_n)_n \longrightarrow x$ means there exists a sequence $(s_n)_n \downarrow x$ and $(t_n)_n \uparrow x$ such that $t_n x_n \leq s_n$for each n. But $(x_n)_n$ is such a $(s_n)_n$, and the constant sequence $(x)_n$ is such a $(t_n)_n$.
q.e.d.

In order to be able to talk about order Cauchy sequences, one should have some notion of distance on L. There are two "standard" ways to measure distance in a LM_m, by dualizing two main logical implicative operations:
• The Heyting distance:

$$d_H(x,y) = (x-y) \vee (y-x) = (x \;\wedge \bigvee_{i\in\{0,\ldots,m-1\}} \varphi_i(x) \wedge \overline{\varphi_i(y)}) \vee (y \;\wedge \bigvee_{i\in\{0,\ldots,m-1\}} \varphi_i(y) \wedge \overline{\varphi_i(x)}) \;;$$

• The pointwise distance ([9]):

$$d_P(x,y) = \bigvee_{i\in\{0,\ldots,m-1\}} (\varphi_i(x) \wedge \overline{\varphi_i(y)}) \vee (\varphi_i(y) \wedge \overline{\varphi_i(x)}) \;.$$

Another "distance" derived from logic using the so called *weak implication* would be

$$d_W(x,y) = (\varphi_{n-1}(x) \wedge \overline{y}) \vee (\varphi_{n-1}(y) \wedge \overline{x}) \;.$$

But this will not fall under our interest in this paper because of two reasons: first, it fails to satisfy a basic and intuitive distance axiom, namely $d(x,x) = 0$; and second, if we still force our way into toplogical aspects of d_W, we obtain nothing more than the Boolean topology on $C(L)$, together with all the other points gathered unseparatedly around their Boolean nuances.

So we have at least two candidates for measuring distance in a LM_m - d_H and d_P. This suggests that distance should be rather axiomatized than fixed, leaving the actual operation as free as possible. Therefore, we define a generic notion of distance in LM_ms

Definition 3.2 A *distance* on L is a binary operation $d : L \times L \longrightarrow L$ with the following four properties (for all $x, y, z, v \in L$, $A \subseteq L$):
D1. $d(x,x) = 0$;
D2. $d(x,y) = d(y,x)$;
D3. $x \leq y \vee d(x,y)$;
D4. $d(d(x,z), d(y,z)) \leq d(x,y)$;
D5. $d(x \vee z, y \vee z) \leq d(x,y)$;
D6. $d(x,y) \leq d(x,z)$ if $x \leq y \leq z$.
D7. if $(x_n)_n \downarrow x$, then $\bigwedge_{n\in I\!N} d(x_n, x) = 0$;
D8. for each $i \in \{0,\ldots,m-1\}$, $d(\varphi_i(x), \varphi_i(y)) \leq \varphi_{m-1}(d(x,y))$.
D9. $d(\overline{x}, \overline{y}) \leq \varphi_{m-1}(d(x,y))$.

The axioms D1-D5 have a classical metrical look, if we notice that, in the discussed structures, "addition of two quantities" is provided by suprema. D6 is stating that the distance is taken "along the order"; D7 connects "closeness by distance" to "closeness by order". Finally, D8 and D9 assert a certain compatibility with the negation and the Chryssipian nuances φ_i.

Lemma 3.2 Let $d : L \times L \longrightarrow L$ be a distance on L. Then:
(1) $d(x,y) = 0$ implies $x = y$;
(2) $d(x,y) \leq d(x,z) \vee d(z,y)$;
(3) $d(x,y) \leq d(u,v)$, if $x, y \in [u,v]$;
(4) $d(x \vee y, u \vee v) \leq d(x,u) \vee d(y,v)$;
(5) $d(d(x,y), d(u,v)) \leq d(x,u) \vee d(y,v)$;
(6) $d(x \wedge y, u \wedge v) \leq \varphi_{m-1}(d(x,u) \vee d(y,v))$;
(7) Assume $(x_n)_n \downarrow$, and, for each n, $x \leq x_n$; then $(x_n)_n \downarrow x$ iff $(d(x_n,x))_n \downarrow 0$;
(8) Assume $(x_n)_n \uparrow$, and, for each n, $x_n \leq x$; then $(x_n)_n \uparrow x$ iff $(d(x_n,x))_n \downarrow 0$;

Proof:
(1): By D3, we have $x \leq y \vee d(y,x) = y \vee 0 = y$ and, similarly, $y \leq x$; hence $x = y$.
(2): By D3 and D4,

$$d(x,y) \leq d(x,z) \vee d(d(x,z), d(x,y)) \leq d(x,z) \vee d(z,y) \ .$$

(3): We use point (2), D6, and D2:

$$d(x,y) \leq d(x,u) \vee d(u,y) \leq d(v,u) \vee d(u,v) = d(u,v) \ .$$

(4): Apply point (2) and D5:

$$d(x \vee y, u \vee v) \leq d(x \vee y, x \vee v) \vee d(x \vee v, u \vee v) \leq d(y,v) \vee d(x,u) \ .$$

(5): Similar to (4), just that we apply D4 instead of D5.
(6): Apply the LM_m properties, point (4), and D9:

$$d(x \wedge y, u \wedge v) = d(\overline{\overline{x} \vee \overline{y}}, \overline{\overline{u} \vee \overline{v}}) \leq \varphi_{m-1}(d(\overline{x} \vee \overline{y}, \overline{u} \vee \overline{v})) \leq$$

$$\leq \varphi_{m-1}(d(\overline{x}, \overline{u}) \vee d(\overline{y}, \overline{v})) \leq \varphi_{m-1}(\varphi_{m-1}(d(x,u)) \vee \varphi_{m-1}(d(y,v))) = \varphi_{m-1}(d(x,u) \vee d(y,v)) \ .$$

(7): "only if" is axiom D7.
"if": By D3, $x_n \leq x \vee d(x, x_n)$; it remains to apply Lemma 2.1.(12) to obtain

$$\bigwedge_n x_n \leq \bigwedge_n (x \vee d(x,x_n)) = x \vee \left(\bigwedge_n d(x,x_n) \right) = x \ .$$

(8): "only if": We have that $(\overline{x_n})_n \downarrow \overline{x}$, hence, by D7, $(d(\overline{x_n}, \overline{x}))_n \downarrow 0$; furthermore, $(\varphi_{m-1}(d(\overline{x_n}, \overline{x})))_n \downarrow 0$, and it remains to apply D9.
"if": By D3, $x \leq x_n \vee d(x,x_n)$, so, by residuation, $x_n \leq x - d(x,x_n)$; apply now Lemma 2.1.(13):

$$\bigvee_n x_n \leq \bigvee_n (x - d(x,x_n)) = x - \left(\bigwedge_n d(x,x_n) \right) = x \ .$$

q.e.d.

Proposition 3.1 d_H and d_P are distances on M.

Proof:
Let us first check the axioms D1-D9 for d_H (remember that $d_H(x,y) = (x-y) \vee (y-x)$ - denote, for the moment, $d = d_H$):
D1: $x - x = 0$, so $d(x,x) = 0$;
D2 follows at once from the commutativity of $\vee$;
D3: We know that $x - y \leq d(x,y)$; it suffices to apply residuation .
D4: By Lemma 2.1.(11), $(x-z)-(y-z) \leq (x-y)$ and $(z-x)-(z-y) \leq (y-x)$; we now apply Lemma 2.1.(15), to obtain:

$$d(x,z) - d(y,z) = [(x-z) \vee (z-x)] - [(y-z) \vee (z-y)] \leq$$

$$\leq [(x-z)-(y-z)] \vee [(z-x)-(z-y)] \leq (x-y) \vee (y-x) = d(x,y) \ .$$

Similarly, $d(y,z) - d(x,z) \leq d(y,x)$. Hence the desired inequality.
D5: By Lemma 2.1.(11), $(x-z) \vee (z-y) \leq (x-y)$ and $(z-x) \vee (y-z) \leq (y-x)$; furthermore, by Lemma 2.1.(15), we obtain:

$$(x \vee z) - (y \vee z) \leq (x-y) \vee (z-z) = x - y \ ;$$

$$(y \vee z) - (x \vee z) \leq (y - x) \vee (z - z) \leq y - x \ .$$

Hence the desired inequality.
D6 follows from the fact that $-$ is decreasing on the second argument;
D7: Assume $(x_n)_n \downarrow x$. Then

$$d(x, x_n) = x_n - x = \bigvee_{i \in \{0,\ldots,m-1\}} [x \wedge \varphi_i(x_n) \wedge \overline{\varphi_i(x)}] \ .$$

Furthermore, for each $i \in \{0, \ldots, m-1\}$, $(\varphi_i(x_n))_n \downarrow \varphi_i(x)$, and hence $[x \wedge \varphi_i(x_n) \wedge \overline{\varphi_i(x)}]_n \downarrow 0$; we now apply the 0-sphere property $m-1$ times.
D8: We have that $\varphi_i(x) - \varphi_i(y) = (\varphi_i(x) \wedge \overline{\varphi_i(y)})$. On the other hand,

$$\varphi_{m-1}(x - y) = \bigvee_{i \in \{0,\ldots,m-1\}} (\varphi_{m-1}(x)\varphi_i(x) \wedge \overline{\varphi_i(y)}) = \bigvee_{i \in \{0,\ldots,m-1\}} (\varphi_i(x) \wedge \overline{\varphi_i(y)}) \ .$$

So $\varphi_i(x) - \varphi_i(y) \leq \varphi_{m-1}(x - y)$; we finally apply the fact that φ_{m-1} commutes with suprema.
D9:

$$d(\overline{x}, \overline{y}) = \bigvee_{i \in \{0,\ldots,m-1\}} \overline{x} \wedge \varphi_i(\overline{x}) \wedge \overline{\varphi_i(\overline{y})} =$$

$$= \bigvee_{i \in \{0,\ldots,m-1\}} \overline{x} \wedge \overline{\varphi_{m-i}(x)} \wedge \overline{\varphi_{m-i}(y)} = \bigvee_{i \in \{0,\ldots,m-1\}} \overline{x} \wedge \overline{\varphi_i(x)} \wedge \overline{\varphi_i(y)} \leq \varphi_{m-1}(y - x) \ .$$

As at the precedent point, we conclude using commutation of φ_{m-1} with $\vee$.

We now check the axioms for d_P. Just until the end of the proof, we denote d_W by d; notice that d measures distance using only values from the Boolean center of L; so $\varphi_i(d(x, y)) = d(x, y)$, for each $i \in \{0, \ldots, m-1\}$.
D1 and D2 follow immediately from the Boolean properties;
D3: We need to check that, for each $i \in \{0, \ldots, m-1\}$, $\varphi_i(x) \leq \varphi_i(y) \vee \varphi_i(d(x, y))$, that is $\varphi_i(x) \leq \varphi_i(y) \vee d(x, y)$, that is $\varphi_i(x) \wedge \overline{\varphi_i(y)} \leq d(x, y)$, which is true;
D4: Because $d(x, z)$ and $d(y, z)$ are Boolean elements, we have that

$$d(d(x, z), d(y, z)) = (d(x, z) \wedge \overline{d(y, z)}) \vee (d(y, z) \wedge \overline{d(x, z)}) \ ,$$

so, by symmetry, it suffices to prove $d(x, z) \wedge \overline{d(y, z)} \leq d(x, y)$, which means $d(x, z) \leq d(x, y) \vee d(y, z)$; the last follows immediately from the fact that, for each $i \in \{0, \ldots, m-1\}$, $\varphi_i(x)$, $\varphi_i(y)$, and $\varphi_i(z)$ are Booleans, so

$$\varphi_i(x) \wedge \overline{\varphi_i(z)} \leq (\varphi_i(x) \wedge \overline{\varphi_i(y)}) \vee (\varphi_i(y) \wedge \overline{\varphi_i(z)}) \ .$$

D5 follows in a similar fashion to D4, using

$$\varphi_i(x \vee z) \wedge \overline{\varphi_i(y \vee z)} = [\varphi_i(x) \vee \varphi_i(z)] \wedge [\overline{\varphi_i(y)} \wedge \overline{\varphi_i(z)}] \leq \varphi_i(x) \wedge \overline{\varphi_i(y)} \ .$$

D6 is obvious.
D7: Assume $(x_n)_n \downarrow x$. Then

$$d(x, x_n) = \bigvee_{i \in \{0,\ldots,m-1\}} [\varphi_i(x_n) \wedge \overline{\varphi_i(x)}] \ .$$

Furthermore, for each $i \in \{0, \ldots, m-1\}$, $(\varphi_i(x_n))_n \downarrow \varphi_i(x)$, and hence $[\varphi_i(x_n) \wedge \overline{\varphi_i(x)}]_n \downarrow 0$; finally, apply the 0-sphere property $m-1$ times.
D8: $d(\varphi_i(x), \varphi_i(y)) \leq d(x, y)$ is an immediate equality;
D9: $d(\overline{x}, \overline{y}) = d(x, y)$.
q.e.d.

Remark 3.1 d_H is the smallest distance (in the sense of $\leq$) on L.

Indeed, by D3, any distance d has the property that $x \leq y \vee d(x, y)$; hence, by residuation, $x - y \leq d(x, y)$; so $d_H(x, y) \leq d(x, y)$.

Notice that, if a distance is provided to L, then the order convergence can be expressed in terms of it.

Proposition 3.2 Let d be a distance on L, $(x_n)_n$ be a sequence from L and $x \in L$. Then $(x_n)_n \longrightarrow x$ iff there exists $(s_n)_n \downarrow 0$ a sequence from L such that, for each $n \in I\!N$, $d(x_n, x) \leq s_n$.

Proof:
"if": Define, for each $n \in I\!N$, $a_n = x - s_n$ and $b_n = x \vee s_n$. Obviously, $a_n \leq x \leq b_n$, $(a_n)_n \uparrow$, and $(b_n)_n \uparrow$; also, by Lemma 2.1.(13 and 12), $(a_n)_n \uparrow x$ and $(b_n)_n \downarrow x$. Finally, using D3 and the residuation, we get

$$x_n \leq x \vee d(x, x_n) \leq x \vee s_n = b_n \ ;$$

$$x \leq x_n \vee d(x, x_n) \leq x_n \vee s_n \ \text{ so } \ a_n = x - s_n \leq x_n \ .$$

"only if": Assume $(x_n)_n \longrightarrow x$, so there exist $(a_n)_n \uparrow x$, $(b_n)_n \downarrow x$ such that, for each n, $a_n \leq x_n \leq b_n$. By D7 and Lemma 3.2.(8), $(d(b_n, x))_n \downarrow 0$ and $(d(a_n, x))_n \downarrow 0$. Define, for each n, $s_n = d(a_n, x) \vee d(b_n, x)$; by the 0-sphere property, $(s_n)_n \downarrow 0$. Also, by Lemma 3.2.(3), since $x_n, x \in [a_n, b_n]$, we get

$$d(x_n, x) \leq d(a_n, b_n) \leq d(a_n, x) \vee d(b_n, x) = s_n \ .$$

q.e.d.

Corollary 3.1 The order topology is T_1-separated on L.

Proof:
Let $d = d_H$. We need to prove unicity of the limit for each sequence $(x_n)_n$. Assume that x and y are two limits. Applying Proposition 3.2, we have $(c_n)_n \downarrow 0$ such that, for all $n \in I\!N$, $d(x_n, x) \leq c_n$; and also $(d_n)_n \downarrow 0$ such that, for all $n \in I\!N$, $d(x_n, y) \leq d_n$; hence

$$d(x, y) \leq d(x, x_n) \vee d(x_n, y) \leq c_n \vee d_n \ .$$

By the 0-sphere property, $(c_n \vee d_n)_n \downarrow 0$, so $d(x, y) = 0$; hence $x = y$.
q.e.d.

In the light of Proposition 3.2, which says that order convergence is the same as "distance convergence", the definition of Cauchy sequence comes naturally:

Definition 3.3 Let d be a distance on L. A sequence $(x_n)_n$ from L is said to be a *d-Cauchy sequence* (or *Cauchy sequence* if d is understood) if there exists $(s_n)_n \downarrow 0$ a sequence from L such that, for each $n, p \in I\!N$, $d(x_n, x_{n+p}) \leq s_n$.

As one should expect, convergence implies the Cauchy property:

Proposition 3.3 Let d be a distance on L, $x \in L$ and $(x_n)_n \subseteq L$, such that $(x_n)_n \longrightarrow x$. Then $(x_n)_n$ is a d-Cauchy sequence.

Proof:
By Proposition 3.2, there exists $(s_n)_n \downarrow 0$ such that, for each n, $d(x_n, x) \leq s_n$. Let $n, p \in \mathbb{N}$. Then

$$d(x_n, x_{n+p}) \leq d(x_n, x) \vee d(x, x_{n+p}) \leq s_n \vee s_{n+p} = s_n \ .$$

q.e.d.

4 Cauchy completion

We want to study d-Cauchy completions w.r.t. the order convergence, for an arbitrary distance d. For this, we consider enriched LM_ms:

Definition 4.1 A *metrical Lukasiewicz-Moisil algebra* (MLM_m for short) is a pair (L, d), where L is a LM_m and d is a distance on L. If (L, d) and (L', d') are two metrical LM_ms, then *a MLM_m morphism* between them is a LM_m morphism $h : L \longrightarrow L'$ such that, for each $x, y \in L$, $d'(h(x), h(y)) = h(d(x, y))$.

The next Lemma characterizes the continuous MLM_m morphisms, showing that, even in this case of a generic distance, like in all classical cases of fixed distances, continuity comes to commutance with countable infima or suprema.

Lemma 4.1 Let $h : (L, d) \longrightarrow (L', d')$ be a MLM_m morphism. Then the following are equivalent:
(1) h is continuous;
(2) for each sequence from L $(s_n)_n \downarrow$, $\bigwedge_n s_n = 0$ implies $\bigwedge_n h(s_n) = 0$.
(3) same as (2), just that we do not ask that $(s_n)_n$ be decreasing.
(4) h commutes with the countable infima;
(5) h commutes with the countable suprema;
In addition, any continuous MLM_m morphism preserves Cauchy sequences.

Proof:
"(3) implies (2)" and "(4) implies (3)" are obvious, and "(4) iff (5)" is well-known and follows from Lemma 2.2.(4).
"(1) implies (2)": Let $(s_n)_n \downarrow 0$. Then $(s_n)_n \longrightarrow 0$, and so, by continuity, $(h(s_n))_n \longrightarrow 0$; on the other hand, $(h(s_n))_n \downarrow$; apply now Lemma 3.1 to get $(h(s_n))_n \downarrow 0$.
"(2) implies (3)": Let $(s_n)_n$ such that $\bigwedge_n s_n = 0$. Define $(a_n)_n$ by $a_n = \bigwedge_{i \in \{1,\ldots,n\}} s_i$. Obviously, $(a_n)_n \downarrow 0$, so $(h(a_n))_n \downarrow 0$. But, for each n, $h(a_n) = \bigwedge_{i \in \{1,\ldots,n\}} h(s_i)$, hence $\bigwedge_n h(s_n) = \bigwedge_n h(a_n) = 0$.
"(2) implies (4)": First, let $(x_n)_n \subseteq L$ and $x \in L$ such that$(x_n)_n \downarrow x$; by Lemma 3.2.(7) (applied twice) and (2), we get, consecutively: $(d(x_n, x))_n \downarrow 0$, $(h(d(x_n, x)))_n \downarrow 0$, $(d'(h(x_n), h(x)))_n \downarrow 0$, $(h(x_n))_n \downarrow h(x)$. So we proved the property (4) for decreasing sequences. Now, if $(a_n)_n \subseteq L$ such that $\bigwedge_n a_n = x$, define $(x_n)_n$ by $x_n = \bigwedge_{i \in \{1,\ldots,n\}} a_i$, and the desired property follows.
"(2) implies (1)": Notice first that, for the order topology discussed here, "h continuous" not only implies, but is actually equivalent to:
"$(x_n)_n \longrightarrow x$ implies $(h(x_n))_n \longrightarrow h(x)$, for each $x \in L$, $(x_n)_n \subseteq L$."

Indeed, if the last holds, then let $T \subseteq L'$ be a closed set w.r.t. the order topology. Let us prove that $h^{-1}(T)$ is also closed; for this, let $(x_n)_n \subseteq h^{-1}(T)$ and $x \in L$ such that $(x_n)_n \longrightarrow x$; then $(h(x_n))_n \longrightarrow h(x)$, so $h(x) \in T$, which means $x \in h^{-1}(T)$.

We now come back to the needed implication. Take $(x_n)_n \longrightarrow x$. Then, by Proposition 3.2, there exists $(s_n)_n \downarrow 0$ such that, for each n, $d(x, x_n) \leq s_n$. Then $d'(h(x_n), h(x)) = h(d(x, x_n)) \leq h(s_n)$, while $(h(s_n))_n \downarrow 0$; thus $(h(x_n))_n \longrightarrow h(x)$ and we are done.

For the last part of the proposition, let $(x_n)_n \subseteq L$ be a Cauchy sequence. Then, from $d(x_n, x_{n+p}) \leq s_n$, we get $d'(h(x_n), h(x_{n+p})) \leq h(s_n)$, and, since $(h(s_n))_n \downarrow 0$, we are done.

q.e.d.

Lemma 4.2 All the MLM_m operations are continuous w.r.t. the order topology. Also, for each MLM_m operation σ of arity k (with $k \in \{1, 2\}$) and each $(x_1^n)_n, \ldots, (x_k^n)_n$ Cauchy sequences, $(\sigma(x_1^n, \ldots, x_k^n))_n$ is also a Cauchy sequence.

Proof:

Let L be a MLM_m and let $x, y \in L$, $(x_n)_n, (y_n)_n \subseteq L$ such that $(x_n)_n \longrightarrow x$ and $(y_n)_n \longrightarrow y$. then there exist $(s_n)_n \downarrow 0$, $(t_n)_n \downarrow 0$, such that, for all n, $d(x_n, x) \leq s_n$ and $d(y_n, y) \leq t_n$. Then $(s_n \vee t_n)_n \downarrow 0$ and $(\varphi_{m-1}(s_n \vee t_n))_n \downarrow 0$. Moreover,

- By Lemma 3.2.(4), $d(x_n \vee y_n, x \vee y) \leq s_n \vee t_n$;
- By Lemma 3.2.(6), $d(x_n \wedge y_n, x \wedge y) \leq \varphi_{m-1}(s_n \vee t_n)$;
- By Lemma 3.2.(5), $d(d(x_n, y_n), d(x, y)) \leq s_n \vee t_n$;
- By D8, $d(\varphi_i(x_n), \varphi_i(x)) \leq \varphi_{m-1}(s_n)$;
- By D9, $d(\overline{x_n}, \overline{x}) \leq \varphi_{m-1}(s_n)$.

A very similar argument proves the second part of this lemma.

q.e.d.

Following the traditional algebraic practice, we shall often (in this section) denote by L the MLM_m (L, d); also, the distance in each MLM_m, be it called L, L', or other, shall be denoted by the same letter, d, the context making clear what algebra is involved. One can easily see that, because of axiom D7, the class of all MLM_ms is not a variety (one can actually see this by examples). Also, one may want to consider not *all the possible distances* in the sense of Definition 3.2, but only *a certain kind of distances*, like d_H and d_P. Therefore all the below discussion will be held using another parameter: a fixed class of MLM_ms, $\mathcal{K}$.

Definition 4.2 Let $L = (L, d)$ be a LM_m. L is said to be *Cauchy complete* if, within it, all Cauchy sequneces are order convergent. A MLM_m embedding $h : L \longrightarrow L'$ between two elements from $\mathcal{K}$ is said to be *a Cauchy completion of L in $\mathcal{K}$* if the following hold:

C1. L' is Cauchy complete;

C2. h is continuous (w.r.t. the order topology);

C3. for each $L"$ from $\mathcal{K}$ which is Cauchy complete and each continuous MLM_m embedding $g : L \longrightarrow L"$, there exists a unique continuous MLM_m embedding $f : L' \longrightarrow L"$ such that $f \circ h = g$.

The class $\mathcal{K}$ is said to have *the Cauchy completion property* if each element from $\mathcal{K}$ has a Cauchy completion in $\mathcal{K}$.

Remark 4.1 If one considers two MLM_m embeddings $h : L \longrightarrow L'$ and $g : L \longrightarrow L"$ to be *isomorphic* provided there exists a MLM_m isomorphism $f : L \longrightarrow L'$ such that $f \circ h = g$, then one can easily see that the Cauchy completion of an element L in $\mathcal{K}$, if exists, is unique up to an isomorphism.

4.1 Some Cauchy completion criteria

For this subsection, we fix two MLM_ms L and L', such that $L \subseteq L'$ and the inclusion is a continuous MLM_m embedding. We shall need a slightly more general notion of order convergence.

Definition 4.3 Let $(z_n)_n \subseteq L'$ and $z \in L'$. Then $(z_n)_n$ is said to *L-converge to z*, denoted $(z_n)_n \longrightarrow_L z$, if there exists a sequence $(s_n)_n \subseteq L$, with $(s_n)_n \downarrow_L 0$, such that, for each n, $d(z_n, z) \leq s_n$. $(z_n)_n$ is said to be an L-Cauchy sequence if there exists $(s_n)_n \subseteq L$, $(s_n)_n \downarrow_L 0$, such that, for each $n, p \in I\!N$, $d(z_n, z_{n+p}) \leq s_n$.

Notice that, because of continuity, in the above definition, $(s_n)_n \downarrow_L 0$ is the same thing as $(s_n)_n \downarrow_{L'} 0$.[1] In a perfectly similar fashion to Proposition 3.3, one can prove that L-convergence implies L-Cauchy property.

Lemma 4.3 Let $(x_n)_n \subseteq L$ be an L-Cauchy sequence. Then there exists $(z_n)_n \subseteq L$ such that:
- $(z_n)_n \downarrow$;
- for each n, $x_n \leq z_n$;
- $(d(x_n, z_n))_n \longrightarrow_L 0$;
- if, for some $x \in L'$, $(x_n)_n \longrightarrow_L x$, then $(z_n)_n \downarrow_{L'} x$ and $(z_n)_n \longrightarrow_L x$;
- for each $x \in L'$, $(x_n)_n \longrightarrow_L x$ iff $(z_n)_n \longrightarrow_L x$.

Proof:
Since $(x_n)_n$ is L-Cauchy, there exists $(s_n)_n \subseteq L$, $(s_n)_n \downarrow_L 0$ such that, for each n, p, $d(x_n, x_{n+p}) \leq s_n$. We define $(z_n)_n$ by $z_n = \bigwedge_{i\in\{1,\ldots,n\}}(x_i \vee s_i)$. $(z_n)_n$ is obviously decreasing. [Notice also that if $(x_n)_n \longrightarrow_L x$ and $(t_n)_n \downarrow_L 0$ is the corresponding sequence, we can take $(s_n \vee t_n)_n$ instead of $(s_n)_n$ (this also L-converges to 0 because of the 0-sphere property), to make sure that $z_n \geq x$.] In addition, for each n and $i \leq n$,

$$x_n \leq x_i \vee d(x_i, x_n) \leq x_i \vee s_i \ ,$$

so $x_n \leq z_n$ for each n. We now want to prove that $(z_n)_n$ is L-Cauchy. Let $n, p \in I\!N$. We apply Lemma 3.2.(6 and 4) and D6:

$$d(z_n, z_{n+p}) = d\left(\bigwedge_{i\in\{1,\ldots,n\}} x_i \vee s_i \ , \bigwedge_{i\in\{1,\ldots,n+p\}} x_i \vee s_i\right) \leq$$

$$\leq \varphi_{m-1}\left(\bigvee_{i\in\{1,\ldots,n\}} d(x_i \vee s_i, x_i \vee s_i)\right) \leq \vee\varphi_{m-1}\left(\bigvee_{i\in\{n+1,\ldots,n+p\}} d(x_n \vee s_n, x_i \vee s_i)\right) =$$

$$= \varphi_{m-1}\left(\bigvee_{i\in\{n+1,\ldots,n+p\}} d(x_n \vee s_n, x_i \vee s_i)\right) \leq$$

$$\leq \varphi_{m-1}\left(\bigvee_{i\in\{n+1,\ldots,n+p\}} d(x_n, x_i) \vee d(s_n, s_i)\right) \leq \varphi_{m-1}(s_n \vee d(s_n, 0)) \ .$$

Now, by D7, the 0-sphere property, and Lemma 2.2.(3), $(\varphi_{m-1}(s_n \vee d(s_n, 0)))_n \downarrow_{L'} 0$, so, by continuity, $(\varphi_{m-1}(s_n \vee d(s_n, 0)))_n \downarrow_L 0$.

Furthermore, $d(z_n, x_n) = z_n - x_n \leq (x_n \vee s_n) - x_n \leq s_n$, so $(d(z_n, x_n)_n \longrightarrow_L 0$.

Suppose now $(x_n)_n \longrightarrow_L x$, and let $(t_n)_n \subseteq L$, $(t_n)_n \downarrow_L 0$, with $d(x_n, x) \leq t_n$ for each n. We have that

$$d(z_n, x) \leq d(z_n, x_n) \vee d(x_n, x) \ ,$$

[1] However, L-convergence (or L-Cauchy property) only implies, but is not equivalent to, L'-convergence (or L'-Cauchy property).

and the fact that $(z_n)_n \longrightarrow_L x$ follows immediately form $(d(z_n, x_n)_n \longrightarrow_L 0$. Hence $(z_n)_n \longrightarrow_{L'} x$ so, by Lemma 3.1 (sine $(z_n)_n \downarrow$ and $z_n \geq x$ for each n), $(z_n)_n \downarrow_{L'} x$.

Finally, "$(z_n)_n \longrightarrow_L x$ implies $(x_n)_n \longrightarrow_L x$" follows similarly to the converse implication from $(d(z_n, x_n)_n \longrightarrow_L 0$ and $d(x_n, x) \leq d(z_n, x_n) \vee d(z_n, x)$.

q.e.d.

Define $Conv(L) = \{x \in L' \,/\, \exists (x_n)_n \subseteq L,\ (x_n)_n \longrightarrow_L x\}$. Consider the following properties:
P1. L' is Cauchy complete.
P2. $L' = Conv(L)$.
P2'. For each $y \in L'$, there exists $(x_n)_n \subseteq L$ such that:
- $(x_n)_n \downarrow_{L'} y$;
- $(x_n)_n \longrightarrow_L y$.

P3. For each $(y_n)_n \subseteq L'$ such that $(y_n)_n \downarrow_{L'} 0$, there exists $(x_n)_n \subseteq L$ such that:
- for each n, $x_n \geq y_n$;
- $(x_n)_n \downarrow_L 0$.

Notice that, by Lemma 4.3, P2 is equivalent to P2'. Sometimes in the literature, for diverse types of structures, the notion of Cauchy completion is expressed in terms of countable suprema-preserving embedding satisfying P1 and P2. However, we feel that Definition 4.2 gives a more acurate expression of "completion". We shall see that P1 and P2 always imply completion, while completion implies P1 and P2 in the most encountered cases, those with $\mathcal{K}$ being closed to sub-models.

One can readily see that P3 is a strong property of "topological compatibility" between L and L', beyond continuity of the embedding - P3, together with continuity, assures us that L-limit is the same thing as L'-limit and L-Cauchy is the same as L'-Cauchy.

Lemma 4.4 P2 implies P3.

Proof:

We shall make use of P2'. Let $(y_n)_n \subseteq L'$ such that $(y_n)_n \downarrow_{L'} 0$. For each n, there exist $(x^n_k)_k \subseteq L$, such that $(x^n_k)_k \downarrow_{L'} y_k$.

Define $z_n = \bigwedge_{i \in \{1,\ldots,n\}} x^n_i$. We have that:
- for each n and $i \leq n$, $y_n \leq y_i \leq x^n_i$, so $y_n \leq z_n$;
- $(z_n)_n \subseteq L$;
- $(z_n)_n \downarrow$, because, for each $n, i \in I\!N$, $x^n_i \geq x^{n+1}_i$.

Furthermore, it is clear that $Lb_{L'}((y_n)_n) = Lb_{L'}((x^k_n)_{n,k}) = Lb_{L'}((z_n)_n)$. Hence $\bigwedge^{L'} z_n = \bigwedge^{L'} y_n = 0$; afortiori, $(z_n)_n \downarrow_L 0$.

q.e.d.

Proposition 4.1 If P1 and P2 hold, then the inclusion $\iota : L \longrightarrow L'$ is the Cauchy completion of L.

Proof:

We only need to check the universality property. Let L' be an element from $\mathcal{K}$ which is Cauchy complete and let $g : L \longrightarrow L'$ be a continuous MLM_m embedding. Define $f : L' \longrightarrow L$" as follows. Let $y \in L'$; by P2, there exists $(x_n)_n \subseteq L$ such that $(x_n)_n \longrightarrow_L y$; hence $(x_n)_n$ is an L-Cauchy sequence, hence $(g((x_n)_n))_n$ is an L"-Cauchy sequnce in L"; but L" is Cauchy complete, so there exists $z \in L$" such that $(g((x_n)_n))_n \longrightarrow_{L"} z$; we put $f(y) = z$.
I. f is well defined. Indeed, if $(x_n)_n, (a_n)_n \subseteq L$ such that $(x_n)_n \longrightarrow_L y$ and $(a_n)_n \longrightarrow_L y$, let $(s_n)_n \downarrow_L 0$ and $(t_n)_n \downarrow_L 0$ be the corresponding convergence sequneces from L. Then

$$d(x_n, a_n) \leq d(x_n, x) \vee d(x, a_n) \leq s_n \vee t_n \ ,$$

where $(s_n \vee t_n)_n \downarrow_L 0$. So $d(g(x_n), g(a_n)) \leq g(s_n \vee t_n)$, and it immediately follows that $(g(x_n))_n$ and $(g(a_n))_n$ have the same L"-limit in L".

II. The function f extends g. Indeed, for each $x \in L$, the constant sequence $(x)_n$ L-converges to x, so $f(x) = g(x)$.

III. f is a MLM_m morphism. This follows by a routine check using separation (Corollary 3.1), the continuity of MLM_m operations, together with the fact that g is a continuous morphism. For example, let us prove that $f(a \vee b) = f(a) \vee f(b)$. We have $(a_n)_n \longrightarrow_L a$, $(b_n)_n \longrightarrow_L b$, with $(a_n)_n, (b_n)_n \subseteq L$. Then $(a_n \vee b_n)_n \longrightarrow_L a \vee b$, so

$$(g(a_n) \vee g(b_n))_n = (g(a_n \vee b_n))_n \longrightarrow_{L"} f(a \vee b) .$$

On the other hand, $(g(a_n) \vee g(b_n))_n \longrightarrow f(a) \vee f(b)$. So $f(a \vee b) = f(a) \vee f(b)$.

IV. f is a MLM_m embedding. Indeed, if $a, b \in L'$ such that $a \neq b$, then $d(a, b) > 0$. By P2', there exist $(a_n)_n, (c_n)_n \subseteq L$ such that:

- $(a_n)_n \longrightarrow_L a$ and, for each n, $a_n \geq a$;
- $(b_n)_n \longrightarrow_L b$ and, for each n, $b_n \geq b$.

Let $(s_n)_n \downarrow_L 0$ and $(t_n)_n \downarrow_L 0$ be the corresponding convergence sequences. Denote $c_n = s_n \vee t_n$; we have $(c_n)_n \downarrow_L 0$, and also, by continuity of ι, $(c_n)_n \downarrow_{L'} 0$. Furthermore,

$$d(a, b) \leq d(a, a_n) \vee d(a_n, b_n) \vee d(b_n, b) \leq d(a_n, b_n) \vee c_n ,$$

so $d(a_n, b_n) \leq d(a, b) - c_n$. If, by absurd, $d(a, b) - c_n = 0$ for each n, then $d(a, b) \leq c_n$ for each n, so $d(a, b) = 0$, which is a contradiction. So there exists $m \in I\!N$ such that $d(a, b) - c_m > 0$; denote $\delta = d(a, b) - c_m$; since $(s_n)_n \downarrow$, $d(a, b) - c_n > \delta$ is true for each $n \geq m$. So, from a certain n, $d(a_n, b_n) \geq \delta > 0$. Now, it is not possible that $\bigwedge_n^L d(a_n, b_n) = 0$, because it would imply, by continuity and Lemma 4.1.(3), that $\bigwedge_n^{L'} d(a_n, b_n) = 0$, a contradiction to $\delta > 0$. So we can actuall consider $\delta \in L$ a non-zero lower bound of $(d(a_n, b_n))_n$. Since g is an embedding, $h(\delta) > 0$ is a lower bound of $(d(g(a_n), g(b_n))_n$. The last immediately implies that $(g(a_n))_n$ and $(g(b_n))_n$ cannot have the same L"-limit in L"; hence $f(a) \neq f(b)$.

V. f is continuous. Indeed, let $(y_n)_n \subseteq L'$, with $(y_n)_n \downarrow_{L'} 0$. By Lemma 4.4, P3 holds, so there exists $(x_n)_n \subseteq L$, with $(x_n)_n \downarrow_L 0$ and $x_n \geq y_n$ for each n. Since g is continuous, $(g(x_n))_n \downarrow_{L"} 0$; moreover, f, being an MLM_m morphism, is increasing, so, for each n, $f(y_n) \leq f(x_n) = g(x_n)$, so $(f(y_n))_n \downarrow_{L"} 0$.

VI. Finally, the unicity of f is assured by P2: if $f' : L' \longrightarrow L$" is another such function, then, for each $y \in L'$, we get $(x_n)_n \subseteq L$ with $(x_n)_n \longrightarrow_L y$; so, by the continuity of f', $(g(x_n))_n = (f'(x_n))_n \longrightarrow_{L"} f'(y)$. Thus $f'(y)$ is the unique L"-limit of $(g(x_n))_n$, hence $f'(y) = f(y)$.

q.e.d.

Proposition 4.2 Assume that $\mathcal{K}$ is closed to submodels. Then:

(1) $Conv(L)$, with induced operations from L', is an element of $\mathcal{K}$.

(2) If P1 and P3 hold, then $Conv(L)$ is Cauchy complete.

(3) The inclusion $\iota : L \longrightarrow L'$ is a Cauchy completion of L iff P1 and P2 hold.

Proof:

(1): All we need to show that $Conv(L)$ is stable to the MLM_m operations. This follows at once from the continuity of MLM_m operations on L (Lemma 4.2).

(2): Let $(y_n)_n \subseteq Conv(L)$ a $Conv(L)$-Cauchy sequence. Then it is also an L'-Cauchy sequence, so, by P1, there exists $z \in L'$ such that $(y_n)_n \longrightarrow_{L'} z$. By Lemma 4.3, we can actually consider that $(y_n)_n \downarrow_{L'} z$. Using P2', for each n, there exists $(x_k^n)_k \subseteq L$, such that:

- $(x_k^n)_k \longrightarrow_L y_n$;
- $(x_k^n)_k \downarrow_{L'} y_n$.

Define, for each n, $z_n = \bigwedge_{i\in\{1,\ldots,n\}} x_i^n$. Since, for each n, $(x_k^n)_k$ has the same lower bonds in L' as y_n, it follows that $(x_k^n)_{k,n}$ has the same lower bounds in L' as $(y_n)_n$; but $(x_k^n)_{k,n}$ has also the same lower bounds in L' as $(z_n)_n$. These imply that $\bigwedge_n^{L'} z_n = \bigwedge_{L'} y_n = z$; thus, by Lemma 3.1, $(z_n)_n \longrightarrow_{L'} z$; but this implies, by P3, $(z_n)_n \longrightarrow_L z$, that is $z \in Conv(L)$. We obtained that z is the L-limit (and, even surer, the $Conv(L)$-limit) of $(z_n)_n$.
(3) The "if" part was already proved by Proposition 4.1.
"only if": Notice that the inclusion $\iota' : L \longrightarrow Conv(L)$ is continuous, because $\iota : L \longrightarrow L'$ is so. Thus, according to points (1) and (2), Proposition 4.1 can be applied to ι' to conclude that ι' is a Cauchy completion of L. But ι is also assumed to be a Cauchy completion of L.

Furthermore, the inclusion $\iota" : Conv(L) \longrightarrow L'$ is continuous. Indeeed, let $(y_n)_n \subseteq Conv(L)$ such that $(y_n)_n \downarrow_{Conv(L)} 0$. Because P3 holds relative to ι' (by points (1) and (2) and Lemma 4.2), there exists $(x_n)_n \subseteq L$, $(x_n)_n \downarrow_L 0$, such that, for each n, $x_n \geq y_n$. Since ι is continuous, $(x_n)_n \downarrow_{L'} 0$; thus $(y_n)_n \downarrow_{L'} 0$ too. One can immediately see that $\iota"$ is the only continuous MLM_m embedding from the universality property of ι' relative to ι (constructed as in the proof of Proposition 4.1). But this is also a MLM_m isomorphism (because the other universal embedding $g : L' \longrightarrow Conv(L)$ has to be inverse, both left and right, to $\iota"$, as one can see by applying the uniqueness property of $\iota"$ to $1_{Conv(L)}$ and of ι to $1_{L'}$). So $\iota"$ is surjective, which means $L' = Conv(L)$. Hence $\iota = \iota'$ satisfies P1 and P2.

q.e.d.

The below corollary shows that, in order to provide a Cauchy completion of an element of a class $\mathcal{K}$ closed to submodels, it suffices to continuously embed it into a Cauchy complete one.

Corollary 4.1 Assume that $\mathcal{K}$ is closed to submodels and P is an element of $\mathcal{K}$. Then P has a Cauchy completion iff P is continuously embedded into a Cauchy complete element P' of $\mathcal{K}$.

Proof:
We apply the above proposition, with $L = P$ and $L' = P'$. $\iota : P \longrightarrow Conv(P)$ is the desired completion.

q.e.d.

4.2 The existence of Cauchy completion

For this subsection, we assume that $\mathcal{K}$ is a variety. (This hypothesis about $\mathcal{K}$, the strongest so far, still covers the most encountered cases of distances, namely the polinomially defined ones - in particular, d_W and d_H.) We also fix L, an element of $\mathcal{K}$. We are going to construct the Cauchy completion of L in a classical fashion, as a quotient of the algebra of Cauchy sequences.

Define $Cauchy(L) = \{(x_n)_n \subseteq L \;/\; (x_n)_n \text{ is } L\text{-Cauchy in } L\}$.

Lemma 4.5 $Cauchy(L)$, with pointwise defined operations, is a MLM_m.

Proof:
We need to show that $Cauchy(L)$ is a stable part of $L^{\mathbb{N}}$. But this is actually shown by the second part of Lemma 4.2.

q.e.d.

On $Cauchy(L)$, define the binary relation $\equiv$ by $(x_n)_n \equiv (y_n)_n$ iff $(d(x_n, y_n))_n \longrightarrow_L 0$ in L.

Lemma 4.6 $\equiv$ is a MLM_m congruence on $Cauchy(L)$ (hence $Cauchy(L)/\equiv$ is a MLM_m).

Proof:

Let us first show $\equiv$ to be an equivalence. Reflexivity and symmetry are obvious, while transitivity follows at once from Lemma 3.2.(2).

For the compatibility with operations, let $(x_n)_n$, $(y_n)_n$, $(a_n)_n$, $(b_n)_n$ be elements of $Cauchy(L)$ such that $(x_n)_n \equiv (a_n)_n$ and $(y_n)_n \equiv (b_n)_n$; and let $(s_n)_n \downarrow_L 0$ and $(t_n)_n \downarrow_L 0$ such that, for each $n \in I\!N$, $d(x_n, a_n) \leq s_n$ and $d(y_n, b_n) \leq t_n$. Then $(s_n \vee t_n)_n \downarrow_L 0$ and $(\varphi_{m-1}(s_n \vee t_n))_n \downarrow_L 0$. Using D8, D9 and Lemma 3.2.(4,5,6), we obtain the followig:

- $d(\varphi_i(x_n), \varphi_i(a_n)) \leq \varphi_{m-1}(s_n)$;
- $d(\overline{x_n}, \overline{a_n}) \leq \varphi_{m-1}(s_n)$;
- $d(x_n \vee y_n, a_n \vee b_n) \leq s_n \vee y_n$;
- $d(d(x_n, y_n), d(a_n, b_n)) \leq s_n \vee y_n$;
- $d(x_n \wedge y_n, a_n \wedge b_n) \leq \varphi_{m-1}(s_n \vee y_n)$.

These immediately imply compatibility.

q.e.d.

Denote $L' = Cauchy(L)/\equiv$ and, for each $(x_n)_n \in Cauchy(L)$, by $cl((x_n)_n)$ its $\equiv$-congruence class (so $cl : Cauchy(L) \longrightarrow L'$ is the factorization MLM_m morphism.) Also, for each $x \in L$, denote by $(x)_n$ the x-constant sequence, which is of course a Cauchy sequence. (Obviously, $x \mapsto (x)_n$ is a MLM_m embedding between L and $Cauchy(L)$.)

Define $I : L \longrightarrow L'$ by $I(x) = cl((x)_n)$.

Lemma 4.7 *I is an MLM_m embedding.*

Proof:

Being a composition of two MLM_m morphisms, I is also an MLM_m morphism. We only need to prove its injectivity. If $I(x) = I(y)$, then there exists $(s_n)_n \downarrow_L 0$ such that, for each n, $d(x, y) \leq s_n$; so $d(x, y) = 0$, hence $x = y$.

q.e.d.

Lemma 4.8 *Let $(x_n)_n, (y_n)_n \in Cauchy(L)$. Then $cl((x_n)_n) \leq cl((y_n)_n)$ in L' iff there exists $(s_n)_n \subseteq L$, $(s_n)_n \downarrow_L 0$, such that, for each n, $x_n \leq y_n \vee s_n$.*

Proof:

"only if": We have that $cl((x_n \wedge y_n)_n) = cl((x_n)_n)$, so $(x_n \wedge y_n)_n \equiv (x_n)_n$. This means the existence of an $(s_n)_n \subseteq L$, $(s_n)_n \downarrow_L 0$, such that, for each n, $d(x_n \wedge y_n, x_n) \leq s_n$. Now, using D3, we get, for each n,

$$x_n \leq d(x_n \wedge y_n, x_n) \vee (x_n \wedge y_n) \leq (x_n \wedge y_n) \vee s_n \ ,$$

so $x_n \leq (x_n \vee s_n) \wedge (y_n \vee s_n)$, so $x_n \leq y_n \vee s_n$.

"if": We need to show $cl((x_n \vee y_n)_n) \leq cl((y_n)_n)$, that is $cl((x_n \vee y_n)_n) \leq cl((y_n)_n)$. For this notice first that $cl((x_n \vee y_n)_n) \leq cl((y_n \vee s_n)_n)$ (because, for each n, $x_n \vee y_n \leq s_n \vee y_n \vee y_n = y_n \vee s_n$). All that is left to show is $cl((y_n \vee s_n)_n) = cl((y_n)_n)$. This follows from

$$d(y_n, y_n \vee s_n) = d(y_n \vee 0, y_n \vee s_n) \leq d(0, s_n) \ ,$$

(by D5), together with $(d(0, s_n))_n \downarrow_L 0$ by D7.

q.e.d.

Lemma 4.9 *$I : L \longrightarrow L'$ is continuous (w.r.t. the order topology).*

Proof:

Let $(x_n)_n \subseteq L$ such that $(x_n)_n \downarrow_L 0$. We want to show $(I(x_n))_n \downarrow_{L'} 0$. For this, let $Y \in L'$ such that, for each n, $Y \leq I(x_n)$. According to Lemma 4.3, we can take $Y = cl((y_k)_k)$, with $(y_k)_k \downarrow$. Let $n \in I\!N$. By the previous lemma, from $cl((y_k)_k) \leq cl((x_n)_k)$, we find the existence of a sequence $(s_k)_k \subseteq L$ with $(s_k)_k \downarrow_L 0$ such that, for each $k \in I\!N$, $y_k \leq x_n \vee s_k$.

Notice that $(y_k)_k \downarrow_L 0$; indeed, if $z \in L$ is a lower bound of $(y_n)_n$, then

$$z \leq \bigwedge_k (x_n \vee s_k) \leq x_n \vee \bigwedge_k s_k = x_n \ .$$

But this happens for each n, so $z = 0$.

Now, $(y_n)_n \downarrow_L 0$ implies $(d(y_n, 0))_n \downarrow_L 0$, so $Y = cl((y_n)_n) = 0$ in L'.

q.e.d.

Lemma 4.10 Let $(y_n)_n \in Cauchy(L)$. Then $(I(y_n))_n \longrightarrow_{I(L)} cl((y_n)_n)$ in L'.

Proof:

We konw that $d(y_n, y_{n+p}) \leq s_n$ for each $n, p \in I\!N$, with $(s_n)_n \subseteq L$, $(s_n)_n \downarrow_L 0$. Then, for each n,

$$d(I(y_n), cl((y_k)_k)) = d(cl((y_n)_k), cl((y_k)_k)) = cl((d(y_n, y_k))_k) \ .$$

If we show $cl(d(y_n, y_k)_k) \leq I(s_n)$, then we are done. But this is true, by Lemma 4.8, if we take $(t_k)_k$ to be: $t_k = s_n$ if $k \leq n$ and $t_k = s_k$ is $k > n$. Obviously, $(t_k)_k \downarrow_L 0$ and, for each k, $d(y_n, y_k) \leq s_n \vee s_k = s_k \vee t_k$; and $(t_k \vee s_k)_k \downarrow_L 0$.

q.e.d.

Lemma 4.11 L' is Cauchy complete.

Proof:

Let $(Y_i)_i$ an L'-Cauchy sequence in L'. By Lemma 4.3, we can take $(Y_i)_i \downarrow$. Also, according to the same lemma, for each $i \in I\!N$, we can take $Y_i = cl((y_i^k)_k)$, with $(I(y_i^k))_k \downarrow_{L'} Y_i$; by Lemma 4.10, $(I(y_i^k))_k \longrightarrow_L Y_i$. Notice that, by Lemma 4.4, we have that "P2 implies P3" relative to the inclusion $\iota : i(L) \longrightarrow L'$ (or, we might say, relative to the embedding $I : L \longrightarrow L'$); and, by Lemma 4.2, P2 holds. So, using P3, we immediately see that, for a sequence $(Z_n)_n$, $I(L)$-convergence is the same thing as L'-convergence, and $I(L)$-Cauchy is the same thing as L'-Cauchy. So we have that:

- there exists $(t_n)_n \subseteq L$, $(t_n)_n \downarrow_L 0$, such that, for each n, p, $d(Y_n, Y_{n+p}) \leq I(t_n)$;
- for each $i \in I\!N$, there exists $(s_k^i)_k \subseteq L$, $(s_k^i)_k \longrightarrow_L 0$, such that, for each k, $d(I(y_i^k), Y_i) \leq I(s_i^k)$.

Define, for each n, $b_n = \bigwedge_{i \in \{1,\ldots,n\}} y_i^n$. It is clear that

$$Lb_{L'}((Y_i)_i) = Lb_{L'}((I(y_i^k))_{i,k}) = Lb_{L'}((I(b_n))_n) \ .$$

We want to show that $(d(I(b_n), Y_n))_n) \longrightarrow_{I(L)} 0$ in L'. Define $(a_n)_n$ by $a_n = \bigwedge_{i \in \{1,\ldots,n\}} (s_i^n \vee t_i)$. For each n and $j \leq n$, by D6 and Lemma 3.2.(2),

$$d(I(b_n), Y_n) = d(\bigwedge_{i \in \{1,\ldots,n\}} I(y_i^n), Y_n) \leq d(I(y_j^n), Y_n) \leq$$

$$\leq d(I(y_j^n), Y_j) \vee d(Y_j, Y_n) \leq I(s_j^n) \vee I(t_j) \ .$$

So $d(I(b_n), Y_n) \leq \bigwedge_{i \in \{1,\ldots,n\}} I(s_i^n) \vee I(t_i) = I(a_n)$. Furthermore, $(a_n)_n \downarrow_L 0$. Indeed, $(a_n)_n \downarrow$ because, for each i, $(s_i^n)_n \downarrow$. Also, let $z \in L$, $z \leq a_n$ for each n. This means, consecutively:

$$\forall n, \ \forall i \in \{1, \ldots, n\}, \ \ z \leq s_i^n \vee t_i \ ;$$

$$\forall i,\ \forall n \geq i,\quad z \leq s_i^n \vee t_i\ ;$$

$$\forall i,\quad z \leq \bigwedge_{n\geq i}(t_i \vee s_i^n) = t_i \vee \bigwedge_{n\geq i} s_i^n = t_i\ .$$

So, for each $i \in I\!N$, $z \leq t_i$; hence $z = 0$. Thus, $(a_n)_n \downarrow_L 0$.

Now, notice that $(I(b_n))_n$ is $I(L)$-Cauchy. Indeed, for each $n, p \in I\!N$,

$$d(I(b_n), I(b_{n+p})) \leq d(I(b_n), Y_n) \vee d(Y_n, Y_{n+p}) \vee d(Y_{n+p}, I(b_{n+p})) \leq I(a_n \vee t_n \vee a_{n+p}) = I(a_n \vee t_n)\ ,$$

and $(a_n \vee t_n)_n \downarrow_L 0$. This means that $(b_n)_n$ is L-Cauchy in L, so, by Lemma 4.(10), $(I(b_n))_n \longrightarrow_L cl((b_n)_n)$ in L'; by continuity, $(I(b_n))_n \longrightarrow_{L'} cl((b_n)_n)$ in L'; . Since $(d(I(b_n), Y_n))_n \longrightarrow_{L'} 0$; it immediately follows that $(Y_n)_n \longrightarrow_{L'} cl((b_n)_n)$. So we have found a limit in L' for $(Y_n)_n$, finishing the proof.

q.e.d.

Proposition 4.3 The embedding $I : L \longrightarrow L'$ is the Cauchy completion of L.

Proof:

According to Lemmas 4.(10) and 4.(11), P1 and P2 hold relative to the continuous MLM_m embedding I. So, by Proposition 4.1, I is the Cauchy completion of L.

q.e.d.

Corollary 4.2 Any LM_m-algebra has a Cauchy completion w.r.t. the order topology and d_H or d_P (or, in other words, the classes of $d_H - LM_m$s and $d_P - LM_m$s have the Cauchy completion property).

5 The relation to Boolean algebras

5.1 Boolean completions as LM_1 completions

It is well-known that LM_m-algebras generalize Boolean algebras in that, for $m = 1$, the LM_m-algebras are in fact Boolean algebras (together with φ_0 the identity). Thus, the results from the previous section particularize for Boolean algebras, with an axiomatical notion of distance,[2] that is a binary operation $d : B \times B \longrightarrow B$ such that D1-D7 and $d(\overline{x}, \overline{y}) = d(x, y)$ hold.

In particular, if $\mathcal{K}$ is the class of all Boolean algebras with distance $d(x, y) = (x \wedge \overline{y}) \vee (y \wedge \overline{x})$, we obtain the completion result from [25]. In the next subsection, we shall consider completions of Boolean algebras w.r.t. the usual distance (which coincides, for LM_1, both to d_H and d_P).

5.2 Completions along adjunction

There exists a tight relationship between LM_m-algebras and Boolean algebras, expressed by a certain type of adjunction between $\mathcal{B}ool e$, the category of Boolean algebras, and $\mathcal{LM}_m$, the category of LM_m-algebras (see [12]). We are going to investigate, for the LM_m distance d_P (defined poinwise by the distance between the nuances hierarchies) and for the classical Boolean distance $d(x, y) = (x \wedge \overline{y}) \vee (y \wedge \overline{x})$ (which coincides with d_P when Booleans algebra are seen as LM_1s) how completions behave when transported along this adjunction. So, from now one, we only discuss order completions w.r.t. the distance $d = d_P$ - notice that here, because d is polinomially defined in terms of the LM_m operations, "MLM_m morphism" means "LM_m morphism".

[2] We are not aware of any treatement of convergence in Boolean algebras w.r.t. to a generic distance.

For each structure R, be it LM_m or Boolean algebra, denote by $\equiv_R$ the congruence relation on $Cauchy(R)$, defined in the previous section, and by R' the quotient algebra $Cauchy(R)/\equiv_R$. According to Proposition 4.3, the embedding $I_R : R \longrightarrow R'$, given by $I_R(x) = cl((x)_n)$ is the Cauchy completion of R.

Define the functors $C : \mathcal{LM}_m \longrightarrow \mathcal{B}oole$ and $D : \mathcal{B}oole \longrightarrow \mathcal{LM}_m$ by:
• C is already defined on objects: $C(L)$ is the Boolean center of L.
• if $h : L_1 \longrightarrow L_2$ is a LM_m morphism, let $C(h) : C(L_1) \longrightarrow C(L_2)$ be the restriction and corestriction of h.
• if B is a Boolean algebra, let $D(B)$ be the subset of B^m consisting of all decreasing vectors $x = (x_1, \ldots, x_m)$ (i.e. such that, for each $i < j$, $x_i \geq x_j$); define the operations $\vee$, $\wedge$, 0, 1 pointwise; the operation $^-$ by $\overline{(x_1, \ldots, x_m)} = (\overline{x_m}, \ldots, \overline{x_1})$; for each $i \in \{0, \ldots, m-1\}$, $\varphi_i(x_1, \ldots, x_m) = (x_{i+1}, \ldots, x_{i+1})$. Then $D(B)$ becomes a LM_m.
• if $f : B_1 \longrightarrow B_2$ is a Boolean morphism, let $D(f) : D(B_1) \longrightarrow D(B_2)$ be defined by

$$D(f)(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n)) \ .$$

Lemma 5.1 [2] C is faithful, T is fully faithful, and T is a right adjoint of C; the unit of the adjunction is $\eta = (\eta_L)_L$, with $\eta_L : L \longrightarrow D(C(L))$ a LM_m embedding, $\eta_L(x) = (\varphi_{m-1}(x), \ldots, \varphi_0(x))$; the counit is $\epsilon = (\epsilon_B)_B$, with $\epsilon_B : C(D(B)) \longrightarrow B$ a Boolean isomorphism, $\epsilon_B(x) = x_1$ for each $x \in C(D(B))$.

The functors C and T
1) preserve products (constructed in a usual, set-theoretical fashion) in the follwwing way:
• $C\left(\Pi_{k\in K} L_k\right) = \Pi_{k\in K} C(L_k)$;
• $D\left(\Pi_{k\in K} B_k\right) \simeq \Pi_{k\in K} D(L_k)$, by $(x_k^1, \ldots, x_k^m)_k \mapsto ((x_k^1)_k, \ldots, (x_k^m)_k)$.
2) preserve surjective morphisms in the following way:
• let R be a congruence on the LM_m L, R_0 be R restricted to $C(L)$ and $cl_L : L \longrightarrow L/R$ be the canonical LM_m factorization morphism; then R_0 is a congruence on $C(L)$ and $C(cl_L) : C(L) \longrightarrow C(L/R) = C(L)/R_0$ is the factorization Boolean morphism of R_0.
• let R_0 be a congruence on the Boolean algebra B, $R \subseteq D(B) \times D(B)$ be R taken pointwise, and $cl_B : B \longrightarrow L/R_0$ be the Boolean factorization morphism; then R is a congruence on $D(B)$ and $D(cl_B) : D(B) \longrightarrow D(B/R_0) = D(B)/R$ is the LM_m factrorization morphism of R.

A LM_m L is called *Post algebra* if the embedding η_L is actually a LM_m isomorphism.[3]

Lemma 5.2 Let L be a LM_m and let $(x_n)_n \subseteq L$, $x \in L$. Then:
(1) if $(x_n)_n \subseteq C(L)$, then $(x_n)_n \downarrow_L 0$ iff $(x_n)_n \downarrow_{C(L)} 0$;
(2) $(x_n)_n$ is Cauchy in L iff, for each $i \in \{0, \ldots, m-1\}$, $(\varphi_i(x_n))_n$ is Cauchy in $C(L)$;
(3) if $(x_n)_n \subseteq C(L)$ is Cauchy, then $(x_n)_n$ is Cauchy in L iff it is Cauchy in $C(L)$; if $(x_n)_n, (y_n)_n \subseteq C(L)$, then $(x_n)_n \equiv_L (y_n)_n$ iff $(x_n)_n \equiv_{C(L)} (y_n)_n$.
(4) If $(x_n)_n \longrightarrow x$ in L iff, for each $i \in \{0, \ldots, m-1\}$, $(\varphi_i(x_n))_n \longrightarrow \varphi_i(x)$ in $C(L)$;
(5) If $(x_n)_n$ is convergent in L, then, for each $i \in \{0, \ldots, m-1\}$, $(\varphi_i(x_n))_n$ is convergent in $C(L)$;
(6) If $m \geq 2$, then the converse of (5) is not necessarily true;
(7) If L is a Post algebra, then the converse of (5) is true.[4]
(8) If L is Cauchy complete, then $C(L)$ is Cauchy complete; if L is a Post algebra, the converse is also true.

Proof:
(1): One implication is obvious. For the other, assume that $(x_n)_n \downarrow 0$ in $C(L)$ and let $x \in L$ be

[3] Actually, such an algebra is polinomially equivalent to a Post algebra in the sense of [11] - see [2], page 165.
[4] Of course, not only Post algebras enjoy this property - it also holds for all LM_ms of finite Boolean center.

a lower bound of $(x_n)_n$; then, for each $i \in \{0,\dots,m-1\}$ and $n \in \mathbb{N}$, $\varphi_i(x) \le \varphi_i(x_n) = x_n$, so $\varphi_i(x) = 0$. Hence, by the determination principle, $x = 0$.
(2): "only if": $(x_n)_n$ is Cauchy in L, then there exists $(y_n)_n \downarrow 0$ in L such that, for each $n, p \in L$, $d(x_n, x_{n+p}) \le y_n$. This implies that, for each $i \in \{0,\dots,m-1\}$, $(\varphi_i(y_n))_n \downarrow 0$ in L, hence in $C(L)$; furthermore, $d(\varphi_i(x_n), \varphi_i(x_{n+p})) \le d(x_n, x_{n+p}) \le y_n$, so, because $d(\varphi_i(x_n), \varphi_i(x_{n+p}))$ is in $C(L)$, it follows that $d(\varphi_i(x_n), \varphi_i(x_{n+p})) \le \varphi_i(y_n)$.
"if": we have, for each $i \in \{0,\dots,m-1\}$, $(s^i_n)_n \downarrow 0$ in $C(L)$ (hence also in L) such that, for each n, p, $d(\varphi_i(x_n), \varphi_i(x_{n+p})) \le s^i_n$. Define, for each n, $y_n = \bigvee_{i\in\{0,\dots,m-1\}} s^i_n$ and, by the 0-sphere property, $(y_n)_n \downarrow 0$ in L; and also $d(x_n, x_{n+p}) \le y_n$.
(3): The first part follws immediately from point (2), since $x \in C(L)$ iff $\varphi_i(x) = x$ for each $i \in \{0,\dots,m-1\}$. So let us prove the second part.
"only if": We have $(s_n)_n \subseteq L$, $(s_n)_n \downarrow_L 0$, such that, for each n, $d(x_n, y_n) \le s_n$. So $d(\varphi_i(x_n), \varphi_i(y_n)) \le d(x_n, y_n) \le s_n$ and, since $d(\varphi_i(x_n), \varphi_i(y_n))$ is Boolean, it follows that $d(\varphi_i(x_n), \varphi_i(y_n)) \le \varphi_i(s_n)$; and each $(\varphi_i(s_n))_n \downarrow_L 0$, so $(\varphi_i(s_n))_n \downarrow_{C(L)} 0$.
"if": If $(s^i_n)_n$ are the sequences corresponding to $(\varphi_i(x_n))_n \equiv_{C(L)} (\varphi_i(y_n))_n$, one can easily see that $(s_n)_n$ defined by $s_n = \bigvee_{i\in\{0,\dots,m-1\}} s^i_n$ brings $(x_n)_n \equiv_L (y_n)_n$.
(4): The proof is very similar to the one of point (2).
(5) follows immediately from (3): if $(x_n)_n \longrightarrow x$ in L then each $(\varphi_i(x_n))_n$ converges to $\varphi_i(x)$ in L.
(6): Assume that $m = 2$. Let $R \subseteq \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$ be defined by

$$R = \{(A,B) \subseteq \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \ / \ A \text{ is infinite and } B = \mathbb{N} \text{ or } A = \emptyset \text{ and } B \text{ is finite}\}.$$

The operations $\wedge$ and $\vee$ are pointwise intersection and union, the lattice 0 and 1 are $(\emptyset,\emptyset)$ and $(\mathbb{N},\mathbb{N})$; $\overline{(A,B)} = (\mathbb{N}\backslash A, \mathbb{N}\backslash A)$; the operators φ_0 and φ_1 are the projections. One can easily see that these operations make R a LM_2. Consider now the sequence $(X_n)_n = ([n,\infty), \mathbb{N})_n \subseteq R$. Then $(\varphi_0(X_n))_n \downarrow \emptyset$, and $(\varphi_1(X_n))_n \downarrow \mathbb{N}$ in $C(L) = \mathcal{P}(\mathbb{N})$, but $(X_n)_n$ is not convergent in L. this example can be easily generalized to a LM_m of an arbitrary $m \ge 2$.
(7): Assume that, for each $i \in \{0,\dots,m-1\}$, $(\varphi_i(x_n))_n \longrightarrow s_i$ in $C(L)$. Because L is a Post algebra, there exists $x \in L$ such that, for each i, $\varphi_i(x) = s_i$. It now remains to apply point (3).
(8): The first part follows at once from (5) and (2), and the second from (7) and (2).

q.e.d.

Proposition 5.1 Let L be a LM_m and B be a Boolean algebra. Then $C(L') \simeq C(L)'$ and $D(B') \simeq D(B)'$.

Proof:

We have that $C(L^{\mathbb{N}}) = C(L)^{\mathbb{N}}$, and, by Lemma 5.2.(3), for each $(x_n)_n, (y_n)_n \in C(L)^{\mathbb{N}}$,
- $(x_n)_n$ is Cauchy in $C(L)$ iff it is Cauchy in L; so $C(Cauchy(L)) = Cauchy(C(L))$;
- $(x_n)_n \equiv (y_n)_n$ in $Cauchy(L)$ iff $(x_n)_n \equiv (y_n)_n$ in L; so $\equiv_{C(L)}$ is the restriction of $\equiv_L$ to $Cauchy(C(L))$, hence

$$C(L') = C(Cauchy(L)/\equiv_L) = C(Cauchy(L))/\equiv_{C(L)} = Cauchy(C(L))/\equiv_{C(L)} = C(L)'.$$

For the second part, let the LM_m isomorphism $f : D(B^{\mathbb{N}}) \longrightarrow D(B)^{\mathbb{N}}$ defined by $f((x^1_n)_n, \dots, (x^m_n)_n) = (x^1_n, \dots, x^m_n)_n$.

We first notice that, if $(z_n)_n = (x^1_n, \dots, x^m_n)_n \subseteq D(B)$, $(z_n)_n$ is Cauchy in $D(B)$ iff, for each $i \in \{1,\dots,m\}$, $(x^i_n)_n$ is Cauchy in B. Indeed, By Lemma 5.2.(2), $(z_n)_n$ is Cauchy in L iff, for each $i \in \{0,\dots,m-1\}$, $(\varphi_i(z_n))_n$ is Cauchy in $C(D(B))$; and the isomorphism $\epsilon_B : C(D(B)) \longrightarrow B$ takes each $\varphi_i(z_n)$ into x^{i+1}_n. So it makes sense to consider the restriction and corestriction of f, the LM_m isomorphism $f_0 : D(Cauchy(B)) \longrightarrow Cauchy(D(B))$.

In a similar fashion, we can prove: for each $(z_n)_n, (t_n)_n \subseteq D(B)$, with $(z_n)_n = (x^1_n, \dots, x^m_n)_n$ and $(t_n)_n = (y^1_n, \dots, y^m_n)_n$, $(z_n)_n \equiv_{D(B)} (t_n)_n$ iff, for each $i \in \{0,\dots,m-1\}$, $(x^i_n)_n \equiv_B (y^i_n)_n$

(we apply the second part of Lemma 5.2.(2), and again the fact that ϵ_B is an isomorpsism and $\epsilon_B(\varphi_i(z_n)) = x_n^{i+1}$). This means that $\equiv_{D(B)}$ is the image by f_0 of pointwise-$\equiv_B$, which implies that $D(B') = D(Cauchy(B))/ \equiv_B$ and $D(B)' = Cauchy(D(B))/ \equiv_{D(B)}$ are isomorphic, by an isomorphism h_B sending $(cl_{\equiv_B}((x_n^1)_n), \ldots, cl_{\equiv_B}((x_n^m)_n))$ to $cl_{\equiv_{D(B)}}((x_n^1, \ldots, x_n^m)_n)$.

q.e.d.

Remark 5.1 The above proposition shows that that the functors C and D commute with the Cauchy completion, reflecting the "pointwise character" of the distance d_P. It also provides another proof of the d_P-completion existence out of Boolean completion, but only for Post agebras - a d_P-completion for the general case cannot be given, as far as we see, simply by the game of nuances. Actually, both Propositions 5.1 and the following 5.2 (the main results of this subsection) hold within a more general acceptance of the notion "pointwise character" - namely, satisfaction of Lemma 5.2.(2 and 4), *acceptance which includes d_H also.*

Next, we characterize the class of all LM_ms (containing the Post algebras) for which completion can be achieved by completion of nuances.

Proposition 5.2 Let L be a LM_m. Then the following are equivalent:
(1) $D(I_{C(L)}) \circ \eta_L : L \longrightarrow D(C(L)')$ is a Cauchy completion of L;
(2) $I_{D(C(L))} \circ \eta_L : L \longrightarrow D(C(L))'$ is a Cauchy completion of L;
(3) For each $z \in D(C(L))$, there exists $(z_n)_n \subseteq L$ such that $(z_n)_n \longrightarrow_{\eta_L(L)} z$.

Proof: Notice that, if $h_{C(L)} : D(C(L)') \longrightarrow D(C(L))'$ is the isomorphism from the proof of Proposition 5.1, then $h_{C(L)} \circ D(I_{C(L)}) = I_{D(C(L))}$. Hence (1) and (2) are equivalent. Denote $f = I_{D(C(L))}$ and $g = \eta_L$.
"(2) implies (3)": We apply Proposition 4.2.(3) to the embedding $f \circ g$ to find that, for each $z \in D(C(L))'$, there exists $(a_n)_n \subseteq L$ such that $(fg(a_n))_n \longrightarrow_{fg(L)} z$. Then, for each $y \in DC(L)$, we find $(a_n)_n \subseteq L$ such that $(fg(a_n))_n \longrightarrow_{fg(L)} f(y)$; hence $(g(a_n))_n \longrightarrow_{g(L)} y$.
"(3) implies (2)": Notice that g is a continuous embedding, because $(x_n)_n \downarrow 0$ in L is equivalent to $(\varphi_i(x_n))_n \downarrow 0$ in $C(L)$ for each i, that is $(g(x_n))_n = (\varphi_0(x_n), \ldots, \varphi_{m-1}(x_n)_n \downarrow 0$ in $DC(L)$. We now check the universality property for $f \circ g$. Let $j : L \longrightarrow L$" be a continuous embedding of L a Cauchy complete LM_m, L". In a perfectly similar way to the proof of Proposition 4.1 (where to construct the embedding $f : L' \longrightarrow L$" we do not use the Cauchy completion of L), we define a continuous embedding $k : DC(L) \longrightarrow L$", unique with the property that $k \circ g = j$. Now, by the universality of f, we find a unique continuous embedding $v : DC(L)' \longrightarrow L$" such that $v \circ f = k$, so $v \circ f \circ g = j$. The fact that v is the unique continuous embedding with $v \circ f \circ g = j$ follows from the previous above unicities (of v with $v \circ f = k$ and of k with $k \circ g = j$.

q.e.d.

5.3 Completion of axled LM_ms

In [26], it is considered a construction of LM_ms starting from Boolean algebras with ideals. It turned out that this construction provides precisely the axled LM_ms (see [2]). Here, we are interested whether this construction commutes with completions. And we shall see that sometimes it does, in a sense specified below.

We again assume that $d = d_P$. Let us first recall the construction from [26], dressed up in a convenient categorical language. Let $IBoole$ be the category whose objects are pairs (B, I) [Boolean algebra - ideal on it], and the morphisms between two objects (B, I) and (B', I') are Boolean morphisms $f : B \longrightarrow B'$ such that $f(I) \subseteq I'$. Define the functor $E : IBoole \longrightarrow LM_m$ as follows:
- on objects, $E(B, I) = \{(x_1, \ldots, x_n) \in D(B) \ / \ \overline{x_n} \wedge x_1 \in I\}$;

- on morphisms, for $f : (B, I) \longrightarrow (B', I')$ in *Boole*, $E(f) : E(B, I) \longrightarrow E(B', I')$ is the restriction and corestriction of $D(f) : D(B) \longrightarrow D(B')$.

Definition 5.1 An ideal I of a Boolean algebra B is called *broad* if, for each countable $X \subseteq B$ such that $\bigwedge X = 0$, it holds that $X \cup I \neq \emptyset$.

Any ideal in a finite Boolean algebra (more generally, in a Boolean algebra such that 1 is an isolated point w.r.t. the order topology) is broad. An ideal is broad iff it includes a vicinity of 1. The next proposition says that E commutes with completions provided the starting ideal is broad.

Proposition 5.3 Let (B, I) in *IBoole* such that I is broad and let $\iota : B \longrightarrow B'$ the Cauchy completion of B (which is also a set-theoretical inclusion). Then $E(\iota) : E(B, I) \longrightarrow E(B', I)$ is the Cauchy completion of $E(B, I)$.

Proof:
Notice first that, since $C(E(B,I)) = C(D(B))$, $C(E(B',I)) = C(D(B'))$ and because of Lemma 5.2.(4) and [Proposition 4.2, Lemma 4.4], the notion of a sequence $(x_n)_n$ converging to an x is independent of the structure considered (be it any one of $E(B,I)$, $D(B)$, $E(B',I)$, $D(B')$), so long as $(x_n)_n$ and x are from inside that structure. In particular, if $(x_n)_n \subseteq D(B)$ and $x \in D(B)$, then $(x_n)_n \longrightarrow_{D(B)} x$ is equivalent to $(x_n)_n \longrightarrow_{E(B,I)} x$, etc. Thus,we shall use freely "$(x_n)_n \longrightarrow x$".

Let us now show that I closed in B'. Let $(x_n)_n \subseteq I$ such that $(x_n)_n \longrightarrow x$. Because of P2, we can assume that $(x_n)_n \subseteq B$. Than there exists $(s_n)_n \subseteq B$, $(s_n)_n \downarrow 0$ [5], such that, for each n, $d(x_n, x) \leq s_n$. Since I is broad, from a certain n, $s_n \in I$. So we can assume $(s_n)_n \subseteq I$. Recall the construction, in Lemma 4.3 (particularized to the Boolean case), of a decreasing $(z_n)_n \subseteq B$ having the same limit as $(x_n)_n$, that is x; because I is an ideal, z_n stays inside I, so $(z_n)_n \subseteq I$. But this implies $x \in I$.

We are now able to check P1 and P2 for $E(\iota) : E(B,I) \longrightarrow E(B',I)$ and conclude, using Proposition 4.1, that it is a Cauchy completion of $E(B,I)$.

P1: Let $(a_n)_n \subseteq E(B',I)$ be a Cauchy sequence. Because $D(B')$ is Cauchy complete, there exists $a \in D(B')$ such that $(a_n)_n \longrightarrow a$. Denote, for each n, $x_n = \varphi_0(a_n)$, $y_n = \varphi_{m-1}(a_n)$; $x = \varphi_0(a)$, $y = \varphi_{m-1}(a)$. We know that $(x_n)_n \longrightarrow x$, $(y_n)_n \in y$, $\overline{x_n} \wedge y_n \in I$ for each n; and we want to show $\overline{x} \wedge y \in I$; but this is true, since the Boolean operations are continuous and I is Cauchy complete in B'.

P2: Let $a \in E(B',I)$. By Propositions 4.2.(3) and 5.1, P2 holds for $D(\iota) : D(B) \longrightarrow D(B')$, so there exists $(a_n)_n \subseteq D(B)$ such that $(a_n)_n \longrightarrow a$. Denote again $x_n = \varphi_0(a_n)$, $y_n = \varphi_{m-1}(a_n)$, $x = \varphi_0(a)$, $y = \varphi_{m-1}(a)$. Then $(x_n)_n \longrightarrow x$, $(y_n)_n \in y$, $\overline{x} \wedge y \in I$. Applying Lemma 4.3 and its dual, we can take $(x_n)_n$ to be decreasing $\geq x$ and $(y_n)_n$ increasing $\leq x$; so $\overline{x_n} \wedge y_n \leq \overline{x} \wedge y$, so $(a_n)_n \subseteq E(B,I)$.

q.e.d.

6 Lukasiewicz distance on proper LM_m-algebras

An important class of LM_m-algebras is the class of *proper* LM_m-algebras defined in [8]. This structures are obtained adding a family of binary operations to the basic type of LM_m-algebras and some conditions for this operations. In [8] is proved that proper LM_m-algebras provide an axiomatization of the m-valued calculus of Lukasiewicz. The category of proper LM_m-algebras is isomorphic with the category of MV_m-algebras defined in [14], which are the structures commonly used as algebraic counterpart of the m-valued Łukasiewicz logic. In this structures we shall consider the *Lukasiewicz distance* and we shall study convergence and Cauchy completions with respect to this distance.

[5] Notice that, by continuity, $(s_n)_n \downarrow_B 0$ iff $(s_n)_n \downarrow_{B'} 0$.

Definition 6.1 [8] A *proper* LM_m-algebra is a structure $(L, \{F_{ij}\}_{(i,j)\in S_m})$, where L is a LM_m-algebra and $\{F_{ij}\}_{(i,j)\in S_m}$ is a family of binary operations on L such that

$$\varphi_k(F_{ij}(x,y)) = \begin{cases} 0, & \text{if } k \le i-j, \\ J_i(x) \wedge J_j(y), & \text{if } k > i-j, \end{cases} \tag{F}$$

for any x, $y \in L$, $k \in \{0, \ldots, m-1\}$ and $(i,j) \in S_m$, where

$$J_i(x) = \varphi_{m-1-i}(x) \wedge \overline{\varphi_{m-2-i}(x)} \text{ for any } i \in \{0, \ldots, m-2\},$$
$$S_m = \{(i,j) : j < i, 2 \le i \le m-2, 0 \le j \le m-4\} \text{ if } m \ge 4 \text{ and}$$
$$S_m = \emptyset \text{ if } m < 4.$$

If we set $T_m = \{(i,j) : j < i, 1 \le i \le m-2, 0 \le j \le m-3\}$ if $m \ge 3$ and $T_m = \emptyset$ if $m < 3$ then, for $m \ge 3$, we extend the definition of F_{ij} for any $(i,j) \in T_m$:

$F_{10}(x,y) = J_2(x) \wedge J_1(y) \wedge \overline{y}$,

$F_{(m-2)(m-3)}(x,y) = J_{m-2}(x) \wedge J_{m-3}(y) \wedge x$.

It is easy to see that F_{ij} satisfy the condition (F) for any $(i,j) \in T_m$.

As a consequence of the determination principle, the family $F_{ij_{(i,j)\in S_m}}$, if exists, it is unique. Hence, saying that L is a proper LM_m-algebra, we shall tacitly understand that the family $F_{ij_{(i,j)\in S_m}}$ is the only possible one.

In [15] it is proved that the category of proper LM_m-algebras is isomorphic to the category of MV_m-algebras. Recall that an MV_m-*algebra* is a structure $(A, \oplus, ^-, 0)$, where $\oplus$ is a binary operation, $^-$ is a unary operation and 0 is a constant such that the following properties hold for any x, $y \in A$:

(M1) $(A, \oplus, 0)$ is an abelian monoid,

(M2) $\overline{\overline{x}} = x$,

(M3) $\overline{0} \oplus x = \overline{0}$,

(M4) $\overline{(\overline{x} \oplus y)} \oplus y = \overline{(\overline{y} \oplus x)} \oplus x$,

(M5) $mx = (m-1)x$,

(M6) $[(jx) \odot (\overline{x} \oplus \overline{((j-1)x)})]^n = 0$,

for any $x \in A$ and $0 < j < m-1$ such that j does not divide $m-1$, where

$$x \odot y = \overline{(\overline{x} \oplus \overline{y})},\ kx = \underbrace{x \oplus \cdots \oplus x}_{k \text{ times}}.$$

The axioms (M1)-(M4) define the notion of MV-*algebra*. The MV-algebras were defined in [6] and they are the algebraic structures which correspond to the ∞-valued Łukasiewicz logic.

Since the basic MV_m-algebra operations can be polinomially defined using the basic operations of proper LM_m-algebra and vice versa, we shall freely use them without making any difference between the two structures. One can see [10, 15, 8, 24] for the detailed definitions.

Hence, in any proper LM_m-algebra we ca define another difference and a corresponding distance:

$$x -_{luk} y = x \odot \overline{y} = x \wedge \overline{y} \wedge (\bigvee_{i=0}^{m-1} (\varphi_i(x) \wedge \overline{\varphi_i(y)})) \wedge \bigwedge_{(i,j)\in T_m} \overline{F_{i,j}(\overline{x}, \overline{y})},$$
$$d_{luk}(x,y) = (x -_{luk} y) \vee (y -_{luk} x).$$

The distance d_{luk} will be called *Lukasiewicz distance*, since this distance is commonly used in the MV-algebra theory. Still, it is not an axiomatic distance in the sense of Definition 3.2, since the residuation condition (D3) for this distance is $x \le y \oplus d_{luk}(x,y)$ for any x and y.[6]

Using d_{luk} we get corresponding notions of d_{luk}-Cauchy sequence and Cauchy completion of a proper LM_m-algebra, in the style of Definitions 3.3 and 4.2.

[6] We shall use the name "axiomatic distance" for distances in the sense of Definition.32.

Remark 6.1 In [13] the convergence in MV-algebras is studied using the distance d_{luk}. The MV-algebra operations are continuous with respect to the convergence defined by d_{luk}. The construction of the Cauchy completion of an MV-algebra L is classical. If $Cauchy(L)$ is the set of all d_{luk}-Cauchy sequences of L, then $Cauchy(L)$ with the pointwise defined operations is an MV-algebra . On $Cauchy(L)$, define the binary relation $\equiv$ by $(x_n)_n \equiv (y_n)_n$ iff $(d_{luk}(x_n, y_n))_n \longrightarrow_{luk} 0$ in L. Hence $\equiv$ is a congruence relation, so we can consider the quotient $L^{'}_{luk} = Cauchy(L)/\equiv$, which is the Cauchy completion of L in the class of MV-algebra. In particular, the embedding $I : L \longrightarrow L^{'}_{luk}$, $I(x) = cl((x)_n)$ is continuous. If L is an MV_m-algebra, then $Cauchy(L)$ and $L^{'}_{luk}$ are obviously MV_m-algebras. In this case, $L^{'}_{luk}$ is the Cauchy completion of L in the class of MV_m-algebras.

If L is a proper LM_m-algebra, then we can consider L as an MV_m-algebra and we get its d_{luk}-Cauchy completion $L^{'}_{luk}$ in the class of MV_m-algebras as in Remark 6.1. Hence, $L^{'}_{luk}$ is a proper LM_m-algebra which is Cauchy complete with respect to the convergence defined by d_{luk}. Due to the categorical isomorphism between MV_m-algebras and proper LM_m-algebras, this actually means that $L^{'}_{luk}$ is the Cauchy completion of L in the class of proper LM_m-algebras with the distance function d_{luk}.

6.1 The relation with other distances

Let L be a proper LM_m-algebra and d an arbitrary axiomatic distance on L.

Lemma 6.1 The following properties hold:
(1) $d_{luk} \leq d$,
(2) if $(x_n)_n$ is a d-Cauchy sequence in L, then $(x_n)_n$ is a d_{luk}-Cauchy sequence.

Proof: (1) It is straightforward that $d_{luk} \leq d_H$. For an arbitrary distance we use Remark 3.1. (2) is a direct consequences of (1).
q.e.d
In the light of the above lemma, the following is obvious:

Lemma 6.2 If L is Cauchy complete w.r.t. d_{luk}, then (L, d) is a Cauchy complete MLM_m-algebra.

In the following, suppose that (L, d) is an MLM_m-algebra such that L is a proper LM_m-algebra and d is, in addition, polinomially defined using the LM_m-algebra operations. Denote by $L^{'}_d$ the Cauchy completion of L w.r.t. d and by $L^{'}_{luk}$ the Cauchy completion of L w.r.t. d_{luk}. Since d is polinomially defined, $(L^{'}_{luk}, d)$ is also a MLM_m-algebra.

Proposition 6.1 Under the above hypothesis, there exists $\iota : (L^{'}_d, d) \longrightarrow (L^{'}_{luk}, d)$ an embedding of MLM_m-algebras.

Proof: Let $I_d : (L, d) \longrightarrow (L^{'}_d, d)$ be the Cauchy completion of (L, d) and $I_{luk} : L \longrightarrow L^{'}_{luk}$ the Cauchy completion of L w.r.t. d_{luk}. Renark that, since I_{luk} is an LM_m-algebra morphism and d is polinomially defined, we get $I_{luk}(d(x, y)) = d(I_{luk}(x), I_{luk}(y))$ for any $x,\ y \in L$. So, $I_{luk} : (L, d) \longrightarrow (L^{'}_{luk}, d)$ is a continuous embedding of MLM_m-algebras. By Lemma 6.2 (2), $(L^{'}_{luk}, d)$ is Cauchy complete. Hence, there exists an unique continuous embedding of MLM_m-algebras $\iota : (L^{'}_d, d) \longrightarrow (L^{'}_{luk}, d)$ such that $\iota \circ I_d = I_{luk}$.
q.e.d.

Remark 6.2 Since d is polinomially defined, the Cauchy completion is constructed in classical fashion, as a quotient of the algebra of Cauchy sequences (see Section 4.2). If $(x_n)_n$ is a d-Cauchy sequence then, by Lemma 6.1 (2), $(x_n)_n$ is a d_{luk}-Cauchy sequence. Let $cl_d((x_n)_n)$ be the class of $(x_n)_n$ in $L^{'}_d$ and $cl_{luk}((x_n)_n)$ be the class of $(x_n)_n$ in $L^{'}_{luk}$. Then it is straightforward to see that,

$$\iota(L_d^{'}) = \{cl_{luk}((x_n)_n)/(x_n)_n \text{ is } d\text{-Cauchy sequence}\}.$$

Hence, the Cauchy completion w.r.t. d can be obtained as a LM_m-subalgebra of the cauchy completion w.r.t. d_{luk}

6.2 The relation with the Boolean center

Let L be a proper LM_m-algebra. We shall establish a connection between the Cauchy completion $L_{luk}^{'}$ and the Cauchy completion of the Boolean center $C(L)$ (this completion will be denoted $C(L)^{'}$). Remark that, whenever b_1, $b_2 \in C(L)$, $d_{luk}(b_1, b_2) = d_B(b_1, b_2) = (b_1 \wedge \overline{b_2}) \vee (b_2 \wedge \overline{b_1})$ is the usual Boolean distance. Hence, whenever we shall talk about convergence, Cauchy sequences and Cauchy completion for $C(L)$ the distance we shall refer to will be the classical one.

Proposition 6.2 For any $x \in L$ and $(x_n)_n \subseteq L$, the following properties hold:
(1) $(x_n)_n$ is a d_{luk}-Cauchy sequence in L iff $(\varphi_i(x_n))_n$ is a Cauchy sequence in $C(L)$ for any $i \in \{0, \ldots, m-1\}$,
(2) $x_n \longrightarrow_{luk} x$ in L iff $\varphi_i(x_n) \longrightarrow \varphi_i(x)$ for any $i \in \{0, \ldots, m-1\}$.

Proof: The "only if" part is a consequence of the following inequality:

$$d_{luk}(x, y) \leq \bigvee_{i=0}^{m-1} d_B(\varphi_i(x), \varphi_i(y)) \text{ for any } x,\ y \in L.$$

Since the unary operation φ_0, …, φ_{m-1} can be expressed pollinomially using the MV-algebra operations, the "if" part of (1) follows by the fact that the set of all Cauchy sequences is closed to the MV-algebra operations (see Remark 6.1). For the "if" part of (2) we use the fact the MV-algebra operations are continuous w.r.t. d_{luk}.
q.e.d.

Lemma 6.3 If L is Cauchy complete w.r.t. d_{luk}, then $C(L)$ is Cauchy complete. If L is a Post algebra of order m, the converse is also true.

Proof: The proof of Lemma 5.2 (8) still holds for d_{luk}.
q.e.d.

Lemma 6.4 $C(L_{luk}^{'}) \simeq C(L)^{'}$.

Proof: The proof is similar with the proof of Proposition 5.1.
q.e.d.

7 Characterizations of Cauchy completions

We have seen that the Cauchy completness of the Boolean center is not a sufficient condition for a LM_m-algebra to be Cauchy complete. In this section we shall provide a necessary and sufficient condition for the Cauchy completeness of a LM_m-algebra using its representation as a chain of Boolean ideals. Finally, we shall prove that for a special class of axiomatic distances the Cauchy completions are isomorphic. In the particular case when we start with a proper LM_m-algebra, this Cauchy completions are also isomorphic with the Cauchy completion w.r.t. the Łukasiewicz distance d_{luk}. We shall firstly recall some preliminary definitions and results.

Let L be an arbitrary LM_m-algebra. If we define

$$\eta_L : L \longrightarrow D(C(L)),\ \ \eta_L(x) = (\varphi_{m-1}(x), \ldots, \varphi_0(x)),$$

then η_L is an embedding of LM_M-algebras. Moreover, η_L is an isomorphism iff L is a Post algebra (see Lemma 5.1). If $b \in C(L)$, we get $\eta_L(b) = (b, \ldots, b)$ and $C(L) \simeq D(C(L)$.

We also consider the auxiliary unary operations J_0, …, J_{m-2} defined as follows:

$$J_i(x) = \varphi_{m-1-i}(x) \wedge \overline{\varphi_{m-2-i}(x)} \text{ for any } i \in \{0, \ldots, m-2\}.$$

and we set $J_{m-1}(x) = \varphi_0(x)$ for any $x \in L$.

Lemma 7.1 [24] In any LM_m-algebra L, the following properties hold:
(1) $\varphi_i(x) = \bigvee_{k=m-1-i}^{m-1} J_k(x)$ for any $i \in \{0, \ldots, m-1\}$,
(2) if $i \neq j$ then $J_i(x) \wedge J_j(x) = 0$,
(3) $\varphi_k(J_i(x) \wedge x) = 0$ if $k \leq m-i-2$ and $\varphi_k(J_i(x) \wedge x) = J_i(x)$ otherwise, for any $k \in \{0, \ldots, m-1\}$ and $i \in \{0, \ldots, m-2\}$,
(4) Determination principle:

$$x = y \text{ iff } J_i(x) = J_i(y) \text{ for any } i \in \{0, \ldots, m-1\}.$$

In [24], a LM_m-algebra is characterized, modulo isomorphism, by a family of Boolean ideals of the Boolean center $C(L)$. To be more precisely, if L is a LM_m-algebra, we define

$$\mathcal{I}(L) = (J_0(L), \ldots, J_{m-2}(L), C(L)),$$

Conversely, if B is a Boolean algebra and I_0, ..., I_{m-2} are ideals of B such that $I_i = I_{m-2-i}$ for any $i \in \{0, \ldots, m-2\}$, we define

$$\mathcal{S}(I_0, \ldots, I_{m-2}, B) = \{x \in D(B) / J_i(x) \in \eta_B(I_i) \text{ for any } i \in \{0, \ldots, m-2\}\},$$

For two such sequences $(I_0, \ldots, I_{m-2}, B)$ and $(J_0, \ldots, J_{m-2}, C)$, we define

$$(I_0, \ldots, I_{m-2}, B) \simeq (J_0, \ldots, J_{m-2}, C)$$

if there exists a Boolean isomorphism $h : B \longrightarrow C$ such that $h(I_i) = J_i$ for any $i \in \{0, \ldots, m-2\}$.

Under the above definitions, the following properties hold:

$$\mathcal{S}(\mathcal{I}(L)) = \eta_L(L) \simeq L,$$
$$\mathcal{I}(\mathcal{S}(I_0, \ldots, I_{m-2}, B)) = (\eta_B(I_1), \ldots, \eta_B(I_{m-2}), C(D(B))) \simeq (I_0, \ldots, I_{m-2}, B).$$

Definition 7.1 If B is a Cauchy complete Boolean algebra, an ideal $I \subseteq B$ is called *Cauchy complete* if

$$[(b_n)_n \subseteq I \text{ Cauchy sequence and } b_n \longrightarrow b] \text{ imply } b \in I.$$

If B is a Boolean algebra and I_0, ..., I_{m-2} are ideals of B, then the sequence $(I_0, \ldots, I_{m-2}, B)$ is called *Cauchy complete* if B is Cauchy complete and I_0, ..., I_{m-2} are Cauchy complete ideals of B.

In the following we suppose that either $d = d_{luk}$, or d is an axiomatic distance such that the following properties hold:
(C0) d is polinomially defined using the LM_m-algebra operations,
(C1) if b_1, $b_2 \in C(L)$ then $d(b_1, b_2) = d_B(b_1, b_2) = (b_1 \wedge \overline{b_2}) \vee (b_2 \wedge \overline{b_1})$,
(C2) $(x_n)_n$ is a d-Cauchy sequence in L iff $(\varphi_i(x_n))_n$ is a Cauchy sequence in $C(L)$ for any $i \in \{0, \ldots, m-1\}$,
(C3) $x_n \longrightarrow_L x$ w.r.t. d iff $\varphi_i(x_n) \longrightarrow \varphi_i(x)$ for any $i \in \{0, \ldots, m-1\}$.

Remark that, for $d = d_{luk}$, we assume that the LM_m-algebras involved are proper LM_m-algebras. We shall first prove a preliminary result.

Lemma 7.2 In an LM_m algebra L the following properties hold for any $x \in L$ and $(x_n)_n \subseteq L$:
(1) $(\varphi_i(x_n))_n$ is a Cauchy sequence for any $i \in \{0, \ldots, m-1\}$ iff $(J_i(x_n))_n$ is a Cauchy sequence for any $i \in \{0, \ldots, m-1\}$,
(2) $\varphi_i(x_n) \longrightarrow x$ for any $i \in \{0, \ldots, m-1\}$ iff $J_i(x_n) \longrightarrow x$ for any $i \in \{0, \ldots, m-1\}$.

Proof: (1) and (2) are consequences of the following relations:

$$d(\varphi_i(x), \varphi_i(y)) \leq \bigvee_{k=m-1-i}^{m-1} d(J_k(x), J_k(y)) \text{ for } k\{0, \ldots, m-1\},$$
$$d(J_i(x), J_i(y)) \leq d(\varphi_{m-1-i}(x), \varphi_{m-1-i}(y)) \vee d(\varphi_{m-2-i}(x), \varphi_{m-2-i}(y)) \text{ for } k\{0, \ldots, m-2\},$$
$$d(J_{m-1}(x), J_{m-1}(y)) = d(\varphi_{m-1}(x), \varphi_{m-1}(y)).$$

q.e.d.

Now, we able to provide a characterization of the Cauchy completeness and of the Cauchy completion of a LM_m-algebra using the properties of the corresponding sequence of Boolean ideals.

Proposition 7.1 If L is a LM_m-algebra, then L is Cauchy complete w.r.t. d iff $\mathcal{I}(L)$ is Cauchy complete.

Proof: "only if": Suppose that L is a LM_m-algebra, Cauchy complete w.r.t. d. Then $C(L)$ is Cauchy complete by (C1). For any $i \in \{0, \ldots, m-2\}$ we have to prove that $J_i(L)$ is Cauchy complete. Let $i \in \{0, \ldots, m-2\}$ and $(b_n)_n \subseteq J_i(L)$ a Cauchy sequence, $b_n \longrightarrow b$. We have to find an element $x \in L$ such that $J_i(x) = b$. For any n we define $z_n = (b_n, \overset{i}{\ldots}, b_n, 0, \ldots, 0) \in D(C(L))$. Hence, it is straightforward that $J_i(z_n) = \eta_L(b_n)$ and $J_j(z_n) = \eta_L(0)$ for $j \neq i$. In consequence, $J_j(z_n) \in \eta_L(J_j(L))$ for any $j \in \{0, \ldots, m-2\}$, so $z_n \in \mathcal{S}(\mathcal{I}(L)) = \eta_L(L)$. For any n, let $x_n \in L$ be the unique element such that $\eta_L(x_n) = z_n$. It follows that $\eta_L(J_i(x_n)) = J_i(z_n) = \eta_L(b_n)$, so $J_i(x_n) = b_n$ for any n. If $j \neq i$, we can similarly prove that $J_j(x_n) = 0$ for any n. In consequence, for any $j \in \{0, \ldots, m-2\}$, the sequence $(J_j(x_n))_n$ is a Cauchy sequence in $C(L)$. By Lemma 7.2 (1) and (C2), $(x_n)_n$ is a Cauchy sequence of L and, by hypothesis, there exists $x \in L$ such that $x_n \longrightarrow_L x$ w.r.t d. By (C3) and Lemma 7.2 (2), $J_i(x_n) \longrightarrow J_i(x)$ in $C(L)$, hence $b_n \longrightarrow J_i(x)$ in $C(L)$. Since we also have $b_n \longrightarrow b$ in $C(L)$, by Corollary 3.1, it follows that $b = J_i(x)$. We proved that $b \in J_i(L)$, so the Boolean interval $J_i(L)$ is Cauchy complete for any $i \in \{0, \ldots, m-2\}$.
"if": Assume that $\mathcal{I}(L)$ is Cauchy complete and let $(x_n)_n \subseteq L$ be a Cauchy sequence in L. By (C2) and Lemma 7.2 (1), $(J_i(x_n))_n$ is a Cauchy sequence in $J_i(L) \subseteq C(L)$ for any $i \in \{0, \ldots, m-1\}$. By hypothesis, there are $b_i \in J_i(L)$ such that $J_i(x_n) \longrightarrow b_i$ for any $i \in \{0, \ldots, m-1\}$. Since the Boolean algebra operations are continuous, using Lemma 7.1 (2), we get $b_i \wedge b_j = 0$ for any $i \neq j$, so $b_i \leq b_j^*$ for any $i \neq j$. For any $i \in \{0, \ldots, m-1\}$, let $y_i \in L$ such that $J_i(y_i) = b_i$. Set

$$x = b_{m-1} \vee \bigvee_{i=0}^{m-2} (y_i \wedge b_i) = b_{m-1} \vee \bigvee_{i=0}^{m-2} (y_i \wedge J_i(y_i)).$$

By Lemma 7.1 (3), we get $\varphi_i(x) = \bigvee_{k=m-1-i}^{m-1} b_k$ for any $i \in \{0, \ldots, m-1\}$. Hence, for $J_{m-1}(x) = \varphi_{m-1}(x) = b_{m-1}$ and $J_i(x) = b_i \wedge b_{i+1}^* \wedge \cdots \wedge b_{m-1}^*$ for $i \in \{0, \ldots, m-2\}$. Since $b_i \leq b_j^*$ for any $i \neq j$, we get $J_i(x) = b_i$ for $i \in \{0, \ldots, m-1\}$. It follows that $J_i(x_n) \longrightarrow J_i(x)$ for any $i \in \{0, \ldots, m-1\}$ so, by Lemma 7.2 (2) and (C3), $x_n \longrightarrow_L x$ w.r.t. d. We proved that any Cauchy sequence in L has a limit in L, so L is Cauchy complete w.r.t d.

q.e.d.

Proposition 7.2 If B is a Boolean algebra and $I_0, \ldots, I_{m-2}$ are ideals of B such that $I_i = I_{m-2-i}$ for any $i \in \{0, \ldots, m-2\}$, then $(I_0, \ldots, I_{m-2}, B)$ is Cauchy complete iff $\mathcal{S}(I_0, \ldots, I_{m-2}, B)$ is Cauchy complete w.r.t. d.

Proof: If we denote $S = \mathcal{S}(I_0, \ldots, I_{m-2}, B)$, then $J_i(S) = \eta_B(I_i) \simeq I_i$ for any $i \in \{0, \ldots, m-2\}$ and $C(S) = C(D(B)) \simeq B$. It is obvious that the Cauchy sequences and the convergence in Boolean algebras are preserved by Boolean algebra morphisms. Hence, $(I_0, \ldots, I_{m-2}, B)$ is Cauchy complete iff $\mathcal{I}(S) = (J_0(S), \ldots, J_{m-2}(S), C(S))$ is Cauchy complete. By Proposition 7.1, it follows that $\mathcal{I}(S)$ is Cauchy complete iff S is Cauchy complete w.r.t. d and the intended result follows.

q.e.d.

Let B be a Boolean algebra and B' its Cauchy completion defined as a quotient of the Boolean algebra of the Cauchy sequences of B(see Section 4.2). For an ideal J of B set

$$J^{'} = \{cl((b_n)_n)/(b_n)_n \subseteq J \text{ is a Cauchy sequence}\}.$$

Proposition 7.3 Under the above assumptions, $J^{'}$ is a Cauchy complete ideal of $B^{'}$.

Proof: One can easily see that $J^{'}$ is an ideal of $B^{'}$. Let $(Y_n)_n$ be a Cauchy sequence in $J^{'}$ such that $(Y_n)_n \longrightarrow Y$ in $B^{'}$. We have to prove that $Y \in J^{'}$. Since $(Y_n)_n \subseteq J^{'}$, for each n, there exists a Cauchy sequence $(y_n^k)_k \subseteq J$ such that $Y_n = cl((y_n^k)_k)$. We firstly remark that any Boolean algebra is a LM_m-algebra and the classical Boolean distance is an axiomatic distance in the sense of Definition 3.2. Hence, Lemma 4.3 and its dual still hold for Boolean algebras with the classical Boolean distance. By the dual of Lemma 4.3, we can take $y_n^k \uparrow$ for any n, $Y_n \uparrow$ and $(Y_n)_n \longrightarrow Y$ in $B^{'}$. If we set $b_n = \bigvee_{i\in\{1,\ldots,n\}} y_i^n$ for any n then one can prove that $(b_n)_n$ is a Cauchy sequence of B and $(Y_n)_n \longrightarrow cl((b_n)_n)$ in $B^{'}$ (using the dual constructions from the proof of Lemma 4.11). Remark that $(b_n)_n \subseteq J$, because $(y_n^k)_k \subseteq J$ for any n. It follows that $cl((b_n)_n) \in J^{'}$. Since the limit of a sequence in $B^{'}$ is unique, we infer that $Y = cl((b_n)_n)$, so $Y \in J^{'}$. Hence we proved that $J^{'}$ is a Cauchy complete ideal of $B^{'}$.

q.e.d.

Proposition 7.4 If L is a LM_m-algebra and $L^{'}$ is the Cauchy completion of L w.r.t. d then

$$J_i(L)^{'} = J_i(L^{'}) \text{ for any } i \in \{0,\ldots,m-1\}.$$

Proof: Suppose $i \in \{0,\ldots,m-1\}$ is an arbitrary element.
"$\subseteq$": If $Y \in J_i(L)^{'}$ then there exists a Cauchy sequence $(b_n)_n \subseteq J_i(L)$ such that $Y = cl((b_n)_n)$. It follows that $b_n = J_i(x_n)$ for any n, where $(x_n)_n \subseteq L$. By (C2) and Lemma 7.2 (1), $(x_n)_n$ is also a Cauchy sequence in L. Hence $cl((x_n)_n) \in L^{'}$ and $J_i(c\, l((x_n)_n)) \in J_i(L^{'})$. Since the LM_m-algebra operations are pointwise defined on Cauchy sequences, we get $J_i(c\, l((x_n)_n)) = c\, l((J_i(x_n))_n) = cl((b_n)_n) = Y$, so $Y \in J_i(L^{'})$.
"$\supseteq$": If $Y \in J_i(L^{'})$, then $Y = J_i(cl((x_n)_n))$ for some Cauchy sequence $(x_n)_n \subseteq L$. We get $Y = cl((J_i(x_n))_n)$. By $(C2)$ and Lemma 7.2 (1), $(J_i(x_n))_n$ is also a Cauchy sequence in L and $(J_i(x_n))_n \subseteq J_i(L)$. It follows that $Y \in J_i(L)^{'}$.

q.e.d.

Remark 7.1 For any LM_m-algebra L we have

$$\mathcal{I}(L^{'}) = (J_1(L_d^{'}),\ldots,J_{m-2}(L^{'}),C(L)^{'}) = (J_1(L)^{'},\ldots,J_{m-2}(L)^{'},C(L)^{'}).$$

Lemma 7.3 Let L be an arbitrary LM_m-algebra and let d_1, d_2 be two axiomatic distances which satisfy (C0)-(C3). Then L is Cauchy complete w.r.t. d_1 iff L is Cauchy complete w.r.t. d_2. If, in addition, L is a proper LM_m-algebra then L is Cauchy complete w.r.t. d_{luk} iff L is Cauchy complete w.r.t. an axiomatic distance d satisfying (C0)-(C3).

Proof: It follows by (C1), (C2) and (C3). *q.e.d.*

In the sequel we shall prove that the Cauchy completions w.r.t. axiomatic distances satisfying (C0)-(C3) coincide. Since such a distance is polinomially defined using the LM_m-algebra operations, it can be considered on any LM_m-algebra. Hence, for such a distance d and for any LM_m-algebra L, the pair (L,d) is a MLM_m-algebra. For a LM_m-algebra L, we denote by $L_d^{'}$ the Cauchy completion of L in the class $\mathcal{K}$ of all LM_m-algebras enowed with the corresponding distance defined by d. If L is a proper LM_m-algebra, we have denoted by $L_{luk}^{'}$ the Cauchy completion of L w.r.t. d_{luk} in the class of all proper LM_m-algebras.

Proposition 7.5 If L is an arbitrary LM_m-algebra and d_1, d_2 are two axiomatic distances which satisfy (C0)-(C3), then

$$L'_{d_1} \simeq L'_{d_2}$$

(as LM_m-algebras, as MLM_m-algebras with the distance d_1 and as MLM_m-algebras with the distance d_2).

Proof: By Remark 7.1 we get

$$\mathcal{I}(L'_{d_1}) = (J_1(L)', \ldots, J_{m-2}(L)', C(L)') = \mathcal{I}(L'_{d_2}).$$

Since $\mathcal{I}$ establish a categorical equivalence, we infer that L'_{d_1} and L'_{d_2} are isomorphic LM_m-algebras. The distance d_1 and d_2 are polinomially defined, so an LM_m-algebra isomorphism commutes with both d_1 and d_2.

q.e.d.

Proposition 7.6 If L is a proper LM_m-algebra and d is an axiomatic distance which satisfy (C0)-(C3), then

$$L'_{luk} \simeq L'_d$$

(as proper LM_m-algebras and as MLM_m-algebras with the distance d)

Proof: As in the proof of Proposition 7.5, it is straightforward that L'_{luk} and L'_d are isomorphic LM_m-algebras. Since the distance d is polinomially defined, they are also isomorphic as MLM_m-algebras with the distance d. We only have to prove that L'_d is a proper LM_m-algebra. Let $h : L'_{luk} \longrightarrow L'_d$ a LM_m-algebra isomorphism and suppose that $\{F_{ij}\}_{(i,j)\in S_m}$ is the unique family of operations that provide a proper LM_m-algebra structure for L'_{luk}. If we define

$$G_{ij}(x, y) = h(F_{ij}(h^{-1}(x), h^{-1}(y))) \text{ for any } (i, j) \in S_m \text{ and } x,\ y \in L'_d,$$

then $\{G_{ij}\}_{(i,j)\in S_m}$ is a family of binary operations on L'_d which obviously satisfies the condition (F) form Definition 6.1. Consequently, $(L'_d, \{G_{ij}\}_{(i,j)\in S_m})$ is a proper LM_m-algebra.

q.e.d.

Corollary 7.1 If L is a LM_m-algebra, then the Cauchy completions w.r.t. d_H and d_P are isomorphic. If, in addition, L is a proper LM_m-algebra, this Cauchy completion are also isomorphic with the Cauchy completion w.r.t. d_{luk}.

Proof: d_H and d_P satisfy (C0)-(C3). *q.e.d.*

Remark 7.2 If L is an MV_m-algebra then we can consider its Cauchy completion in the class of MV_m-algebras (as in Remark 6.1) or we can consider its Cauchy completion in the class of all MLM_m-algebras with an axiomatic distance d which satisfy (C0)-(C3). The above results asserts that Cauchy completions we get in each case are isomorphic as MV_m-algebras and as MLM_m-algebras.

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